Problem Set n° 2 MAT 883: Knot Theory Due September 18, 2015

(1) Write a paragraph-long executive summary of the first proof we gave that every knot in S^3 bounds a Seifert surface. When you write it, imagine that you will have to reproduce the full proof from your summary in a year. Which details do you want to include, and which would you be able to fill in?
(2) Show that if L is an oriented link in an integer homology sphere, then L bounds a Seifert surface. Your argument should just highlight how to adapt the proof for the case of a knot to that of a link.
(3) *The "half lives, half dies" principle.* Prove that if M is a compact, orientable 3-manifold with boundary, then the kernel of the inclusion-induced map $i_* : H_1(\partial M; \mathbb{Q}) \to H_1(M; \mathbb{Q})$ is a half-dimensional summand of the domain.
(4) A rational Seifert surface for a knot K in a smooth, closed 3-manifold is a nice (smooth, compact, oriented, properly embedded) surface $F \subset X_K$ whose boundary ∂F consists of coherently oriented, essential, simple closed curves on ∂X_K. Prove that every knot K admits a rational Seifert surface. Again, just adapt the proof for the case of $K \subset S^3$, and highlight the differences. What can you say about $[\partial F] \in H_1(\partial X_K)$?
(5) On your first homework, you showed that when $\{p, q\} = \{3, 5\}$, the knot group of the torus knot $T(p, q)$ is isomorphic to $G(p, q) = \langle x, y \,|\, x^p = y^q \rangle$.
  (a) Make sure that your proof adapts to all pairs of relatively prime integers p and q.
  (b) Let Z denote the subgroup of $G(p, q)$ generated by x^p. Prove that Z is contained in the center $Z(G(p, q))$ and that the quotient $G(p, q)/Z$ is isomorphic to the free product $C_p * C_q$.
  (c) Show that the center of $C_p * C_q$ is trivial and conclude that $Z = Z(G(p, q))$. (Hint: each non-identity element in the free product $A * B$ is uniquely represented by a word $c_1 \cdots c_k$, where the c_i are non-identity elements chosen alternately from A and B.)
  (d) Prove that the isomorphism type of $C_p * C_q$ determines the pair $\{p, q\}$. (Hint: consider its maximal finite subgroups.)
  (e) Conclude that the isotopy type of $T(p, q)$ determines the pair $\{p, q\}$. What about the converse?
(6) How many components does the pretzel link $P(p, q, r)$ have? Determine the topological type of the surface that results from applying Seifert's algorithm to its standard diagram. How does it depend on the link orientation? When does it have genus 1?
(7) Prove that if D is a non-trivial knot diagram, then one of the checkerboard surfaces associated with D is non-orientable. Is the assertion still true if "knot" is replaced by "link"?
(8) Given a diagram D and a checkerboard surface F, we form its *Tait graph* $G(F)$ by placing a vertex in each region and an edge for each crossing where two regions touch.
  (a) Characterize when F is orientable in terms of $G(F)$.
  (b) Show that F is orientable if and only Seifert's algorithm outputs it when applied to D.