MT883: Knot theory
Class summaries

The first two classes were each 50 minutes long, and the remaining classes were each 75 minutes long.

1 Aug. 31

Introduction. Basic definitions and examples of knots, links, and equivalence. Themes in knot theory, some of which we'll address. Algebraic-topological invariants: the Alexander polynomial, the Seifert form, the signature, etc. Geometric invariants, like the (hyperbolic) geometry of the complement. Quantum-inspired invariants: Jones polynomial and Vassiliev invariants. The volume conjecture. Knots and braids. Categorification: knot Floer homology, Khovanov homology, and the like. Algorithmic questions, like unknot recognition.

2 Sept. 2

The homology of a link complement. A first glimpse of the infinite cyclic cover of a knot complement.

3 Sept. 9

Existence of Seifert surfaces, Part I. Reference: Saveliev p. 83 and distributed notes.

4 Sept. 11

Existence of Seifert surfaces, Part II. The generality of the method from Lecture 3: every Seifert surface arises in this way. Seifert's algorithm. Statement without proof that it does not yield all Seifert surfaces. Tait colorings and checkerboard surfaces. Reference: Cromwell, pp. 102-104.

5 Sept. 14

Surprise: if F is output by Seifert's algorithm applied to a diagram D and you sum on an unknotted tube, then the result is also output by Seifert's algorithm. Idea: take connected sum of the diagram D with Siddhi's "trefail" diagram of the unknot, or better yet, the "figure ain't" diagram of the unknot.

Linking numbers lk, distinguishing link types, the Seifert pairing θ and Seifert matrices V. Reference: Rolfsen, Chapter 5, Section D.

6 Sept. 16

The construction of the infinite cyclic cover $X_\infty \to X_K$ using the exterior X_F of a Seifert surface F for K. Handle decompositions of surfaces, bases of curves, and dual bases of curves in a surface exterior with respect to the linking pairing. Lemma that V and V^T present the linking pairings between positive and negative push-offs with respect to dual bases. Reference: Lickorish, Chapter 6, pp. 51-52.

7 Sept. 21

Presentations of modules over commutative rings with 1, elementary ideals, modules over $\Lambda = \mathbb{Z}[t, t^{-1}]$, Alexander polynomials, and $H_1(X_\infty)$ as a Λ-module. Reference: Lickorish, Chapter 6, pp. 49-51.

8 Sept. 23

Proof that $tV - V^T$ is a presentation matrix for $H_1(X_\infty)$. Reference: Lickorish, Chapter 6, Theorem 6.5, pp. 54-55.

9 Sept. 28

Calculations of the Alexander polynomial. Knots distinguished by their elementary ideals but not their Alexander polynomials: the Stevedore knot and $P(3, -3, 3)$. Properties of the Alexander polynomial: evaluation at 1, symmetry, behavior under mirroring, orientation reversal, connected sum, and genus bounds. Definition of fibered knots. Fibering the unknot. Beginning to fiber the trefoil knot. References: Crowell and Fox, (4.5) & (4.6), pp. 127-130; Lickorish, Chapter 6, pp. 57-60.

10 Sept. 30

Fibering the complement of the trefoil knot. Product disks, sutured manifolds, and proof of Gabai's characterization of product sutured manifolds. Disk decompositions and the statement (without proof) of Gabai's theorem that disk decomposable Seifert surfaces have minimum genus. Reference: Gabai, *Detecting Fibred Links in* S^3, Comm. Math. Helv. 61 (1986), 519-555, Section 1.

11 Oct. 5

Fibered knots, the infinite cyclic cover, the commutator subgroup of the knot group, and the Alexander polynomial. Reference: Saveliev pp. 87-90.

12 Oct. 7

The Schoenflies problem, Alexander's theorem, (in)compressibility, incompressibility of a minimum genus Seifert surface, cancelability of braids, non-cancelability of knots and the Mazur swindle, an incompressible Seifert surfaces for a connected sum is a boundary sum of incompressible Seifert surfaces for the summands, additivity of the knot genus.