MT855: Combinatorial methods in knot theory

Professor Josh Greene
Boston College
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1 Foundations

What is a knot? Here is a sensible first definition.

Definition 1.1 (Topological Knot) *A knot is a continuous, one-to-one map* $\gamma : S^1 \to S^3$.

When should we consider two knots equivalent? Here again is a sensible first definition.

Definition 1.2 (Topological Isotopy) *A* topological isotopy *from* γ_0 *to* γ_1 *is a continuous map* $i : I \times S^1 \to S^3$, $I = [0, 1]$ *such that* $i(0, -) = \gamma_0$, $i(1, -) = \gamma_1$, *and* $i(t, -) = \gamma_t$ *is a knot for all* $t \in I$.

However, there is an issue with topological isotopy. For example, imagine a trefoil knot in S^3, where the "trefoil part" is contained in a small region of the knot. Pull the two ends of it tight until knotted portion shrinks to a single point, resulting in an unknotted curve in S^3. This means that a trefoil knot is equivalent to the unknot under Definition 1.2.

In order to rectify the situation, the target of the knot γ should play a greater role in the definition of equivalence. Two possibilities come to mind.

Definition 1.3 (Homeomorphism) *A* homeomorphism *between* γ_0 *and* γ_1 *is an orientation-preserving homeomorphism* $h : S^3 \to S^3$ *such that* $h \circ \gamma_0 = \gamma_1$.

We insist that h preserve orientation, since otherwise we get the equivalence of, for instance, the left- and right-hand trefoil knots, which are not apparently equivalent in "real life." To further capture this sense of reality, we should consider the motion of space as we transform one position of a knot into another.

Definition 1.4 (Topological Ambient Isotopy) *A* topological ambient isotopy *carrying* γ_0 *to* γ_1 *is a continuous map* $i : I \times S^3 \to S^3$ *such that* $i(t, -)$ *is a homeomorphism of* S^3 *for all* $t \in I$, $i(0, -)$ *is the identity map, and* $i(1, -) \circ \gamma_0 = \gamma_1$.

In particular, $i(t, -) \circ \gamma_0$ defines an isotopy from γ_0 to γ_1, and $i(1, -)$ is a homeomorphism from γ_0 to γ_1. Thus, this notion of equivalence is at least as strong as both of Definitions 1.2 and 1.3. One feature of Definition 1.3 is that the map h restricts to an orientation-preserving homeomorphism of knot complements $S^3 \setminus \mathrm{im}(\gamma_0) \xrightarrow{\sim} S^3 \setminus \mathrm{im}(\gamma_1)$. As we shall see, the homeomorphism type of the knot complement distinguishes a trefoil knot from the unknot, so Definitions 1.3 and 1.4 give non-trivial notions of equivalence. In fact, these two definitions give *equivalent* notions of knot equivalence, according to the following result.

Theorem 1.5 (Fisher (1960)) *An orientation-preserving homeomorphism of S^3 can be connected to the identity map by a topological isotopy.*

Thus, given a homeomorphism h as in Definition 1.3, we apply Theorem 1.5 to connect the identity map to h by a topological ambient isotopy of S^3 that carries γ_0 onto γ_1.

However, the topological setting is a bit too broad for our purposes: it allows for *wild* knots which, while fascinating, we will leave aside during this course. Henceforth, we will work in the **smooth category**, which leads us to revise the preceding definitions in the expected manner.

Definition 1.6 (Smooth Knot, First Version) *A smooth knot is a smooth embedding $\gamma : S^1 \to S^3$.*

In particular, the derivative γ' never vanishes. For smooth knots we have the following.

Definition 1.7 (Smooth Isotopy) *A* smooth isotopy *from γ_0 to γ_1 is a smooth map $i : I \times S^1 \to S^3$ such that $i(0,-) = \gamma_0$, $i(1,-) = \gamma_1$, and $i(t,-) = \gamma_t$ is a knot for all $t \in I$.*

Definition 1.8 (Diffeomorphism) *A* diffeomorphism *between γ_0 and γ_1 is an orientation-preserving diffeomorphism $h : S^3 \to S^3$ such that $h \circ \gamma_0 = \gamma_1$.*

Definition 1.9 (Smooth Ambient Isotopy) *A* smooth ambient isotopy *carrying γ_0 to γ_1 is a smooth map $i : I \times S^3 \to S^3$ such that $i(t,-)$ is a diffeomorphism of S^3 for all $t \in I$, $i(0,-)$ is the identity map, and $i(1,-) \circ \gamma_0 = \gamma_1$.*

As in the topological setting, it is clear that Definition 1.9 is at least as strong as Definitions 1.7 and 1.8. Remarkably, all three give equivalent notions of equivalence for smooth knots. This follows from a couple of facts. The first is the *isotopy extension theorem*.[1] Thom was the first to establish results of this kind, and the version we state here follows directly from Hirsch's *Differential Topology*, Chapter 8, Theorem 1.3. All manifolds and maps are assumed smooth here and throughout.

Theorem 1.10 (Isotopy Extension Theorem) *If $V \subset M$ is a compact submanifold of a closed manifold and $F : V \times I \to M$ an isotopy of V, then F extends to an ambient isotopy of M.*

Taking V to be (the image of) a knot and M the three-sphere shows that Definition 1.7 yields the same notion of knot equivalence as Definition 1.9. It is instructive to consider what distinguishes the continuous and smooth settings at this point. For example, why does the shrinking trick not show that the trefoil and unknot are equivalent in the smooth setting? The issue concerns the derivative at

[1] Thanks to Nick Wadleigh for reminding us about it.

the shrinkage point $p \in S^1$. The derivative of γ_t' spins around in an S^1-worth of directions in shrinking neighborhoods of p as $t \to 1$. Since $\gamma_1'(p) \neq 0$, these derivatives do not converge, violating smoothness. To relate Definitions 1.8 and 1.9, we apply the smooth analogue to Theorem 1.5.

Theorem 1.11 (Cerf (1968)) *An orientation-preserving diffeomorphism of S^3 can be connected to the identity map by a smooth isotopy.*

In fact, more is true: Hatcher proved a conjecture of Smale by showing that this group nicely deform retracts onto $SO(4)$. As a result of Theorem 1.11, given two knots that are equivalent via a diffeomorphism h in Definition 1.8, we can connect h to the identity map and thereby deduce that the knots are equivalent by Definition 1.9.

Lastly, we ought to pay attention to parameterizations of the domain S^1. If two knots γ_0, γ_1 have the same image, then $\gamma_1 = \gamma_0 \circ h$ for some diffeomorphism h of S^1. The (much easier) 1-dimensional analogue of Cerf's theorem asserts that if h preserves orientation, then there exists an ambient isotopy of S^1 connecting the identity map to h (the relevant version in the topological category holds as well). It follows that γ_0 and γ_1 are smoothly isotopic. The other possibility is that h reverses orientation, in which case γ_0 and $\gamma_1 \circ r$ are isotopic for a reflection r of S^1. Thus, we will settle on the following final definition of a knot.

Definition 1.12 (Smooth Knot, Final Version) *A smooth (un)oriented knot is the image of a smooth embedding $\gamma : S^1 \to S^3$ taken with(out) orientation.*

Definition 1.13 (Smooth Isotopy, Final Version) *Two smooth oriented knots are isotopic if some (any) defining maps for them are isotopic by Definition 1.7. Two smooth unoriented knots are isotopic if some oriented versions of the knots are isotopic.*

As a side remark, an oriented knot is *invertible* if it is isotopic to itself with the opposite orientation and *non-invertible* otherwise. For example, rotating the standard diagram of an oriented trefoil knot about an axis in the projection plane shows that it is invertible. One expects that an arbitrary oriented knot stands a low chance of being invertible, but actually rigorously demonstrating the existence of a non-invertible knot is tricky. The first examples were due to Trotter (1964).[2]

The last important fact we will want relating the smooth and topological categories is the following fundamental result of Moise.

Theorem 1.14 (Moise (1952)) *Every topological 3-manifold admits a unique smooth and piecewise-linear structure.*

[2]Trotter told me that he was working on operator theory and needed a diversion, so he sat in on a knot theory course taught by Ralph Fox. Fox posed the open problem to exhibit a pair of non-invertible knots, and Trotter solved it by an application of the knot group, a topic we will pursue at length. Livingston showed me a nice proof involving cyclic branched covers.

This result allows us to transition between the perspectives provided by these three categories. For example, we commonly refer to homeomorphisms between *a priori* smooth 3-manifolds without any loss.

One handy fact that the smooth setting provides is the *tubular neighborhood* theorem. (Again, you can consult Hirsch's book for a statement and proof: see Chapter 4, Theorem 5.2) This result tells us that a knot is contained in a neighborhood diffeomorphic to a D^2-bundle over S^1. With a little thought (using the orientation on S^3), it follows that this neighbohood is diffeomorphic to a solid torus.

Having settled on a reasonable definition of a knot, the question arises: how do we tell two knots apart? As we already noted, if two knots are isotopic, then their complements are homeomorphic. Thus, we might have success at distinguishing knot types if we have a successful method for distinguishing their complements. How can we do this? The first thought is to use ordinary homology. However, ordinary homology is maximally poor at distinguishing knots, due to the following result that we will establish before long.

Proposition 1.15 *If $K \subset S^3$ is a knot, then*

$$H_*(S^3 \setminus K) = \begin{cases} \mathbb{Z}, & * = 0, 1 \\ 0, & * \geq 2. \end{cases}$$

We suppress the coefficient system whenever we use ordinary homology groups (i.e., with $\mathbb{Z}$ coefficients). The proof of Proposition 1.15 is an application of the Mayer-Vietoris sequence. In spite of the fact that homology does not distinguish knot complements, it nevertheless is a very useful calculation tool. I think that is an important piece of philosophy: if you get a "boring" or "trivial" answer for some invariant, then you can probably leverage it in a significant way as a calculational tool or a way to define more interesting invariants. For example, the simplicity of the homology of a knot complement will allow us to determine the effect of Dehn surgery on homology, build Seifert surfaces, and define the Alexander polynomial. As a more sophisticated example, Lee's perturbation of the Khovanov differential leads to a theory which assigns the group $\mathbb{Q}$ to any knot, but by leveraging this fact, Rasmussen was able to define the s-invariant in Khovanov homology, which has some *s*ensational properties.

The second thought for distinguishing knot complements is to use the fundamental group. This turns out to be an excellent way to distinguish knot types, and we will pursue this approach shortly. One wonders to what extent the knot complement effectively distinguishes knot types. The question of whether non-isotopic knots in S^3 can have orientation-preserving homeomorphic complements is known as the *knot complement problem*, and it was raised by Tietze (1908). A centerpiece of the course will be to establish the solution to this problem.

Theorem 1.16 (Gordon-Luecke (1989)) *Two knots in S^3 are isotopic iff their complements are orientation-preserving homeomorphic.*

We will follow Gordon-Luecke's solution to this problem, which at its core involves some beautiful combinatorics and graph theory. These techniques shed light on several related problems for which we will develop an appreciation over the course of the semester, and I believe they will continue to bear fruit in this area. We mention one immediate corollary to this result, which relies on fundamental work of Waldhausen, et al.

Theorem 1.17 *The fundamental group of a prime knot complement is a complete invariant of its isotopy type up to mirror image.*

We will state a more precise version of this result once we have developed a bit more about the knot group. (We will also define what "prime" and "mirror image" mean very soon.) Thus, whenever one studies a knot invariant, it is (more or less) secretly an invariant of the *knot group*, i.e., $\pi_1(S^3 \setminus K)$. Some knot invariants (like the Jones polynomial or knot Floer homology) are not defined in terms of the knot group, and it is intriguing to pursue the hidden relationship that exists between them.

In light of Theorem 1.17, Andrew posed the following question at the end of class: are compact, 3-dimensional homology solid tori (compact 3-manifolds with the homology groups listed in Proposition 1.15) determined by their fundamental groups? We will be able to answer this question before long.

For more foundational material about knot theory, I highly recommend the following sources:

- Rolfsen, *Knots and links*, Chapters 1-4
- Gordon, *Some aspects of classical knot theory* (in Springer Lecture Notes 685), Sections 0-2
- Crowell and Fox, *Introduction to knot theory*, Chapter 1

For more foundational material about topological, smooth, and PL manifolds, I found this book:

- Kirby and Siebenmann, *Foundational essays on topological manifolds, smoothing, and triangulations*

For background on differential topology, there are the standard sources:

- Hirsch, *Differential topology*
- Guilleman and Pollack, *Differential topology*