12 Coverings and colorings, part I

Deep results to come will show that the homeomorphism type of $X_K$ is a complete invariant of the knot type $K$ and that $\pi(K)$ is nearly so (it is with its peripheral structure). So both provide sensitive tools for studying knots, but the fact that they recover the knot type completely means that they are themselves complicated to understand directly in most cases. This motivates us to find effective means of extracting information from them. One way is to look at representations of $\pi_1(X_K)$. Doing so enabled us to distinguish some small crossing knots. Another is to look at Dehn fillings of $X_K$ (synonymous with Dehn surgeries along $K$). Doing so enabled us to distinguish the right- and left-hand trefoils. Dehn filling is one natural way to create closed 3-manifolds which capture useful information about $K$. Another way is to form cyclic branched coverings, to which we now turn.

12.1 Coverings.

Since $ab(\pi_1(X_K)) \cong C_\infty$, there exists a unique normal subgroup $\pi_n(K) \subset \pi_1(X_K)$ with quotient group $C_n$ for all $n \in \{1, 2, \ldots, \infty\}$:

$$1 \to \pi_n(K) \to \pi_1(X_K) \to C_n \to 1. \tag{1}$$

Associated to $\pi_n(K)$ is a regular $n$-fold cyclic covering $p_n : X^n_K \to X_K$. Its group of deck transformations is $C_n$, and $(p_n)_*$ maps $\pi_1(X^n_K)$ isomorphically onto $\pi_n(K)$. In particular, $\pi_1(X^n_K) \cong \pi_\infty(K) \cong [\pi(K), \pi(K)]$ is the commutator subgroup of $\pi(K)$, and it is contained in $\pi_n(K)$ for all $n$. We will make a simple, natural modification of the spaces $X^n_K$, $n < \infty$, to produce branched covers associated to $K$.

First, let us examine what happens to the meridian and longitude under the covering maps. For concreteness, we place a basepoint on $\partial X_K$, choose a meridian $\mu$ and Seifert longitude $\lambda$ that pass through it, and fix a lift of the basepoint in each cover $X^n_K$. Suppose that $n < \infty$. Let $\mu_n \subset \partial X^n_K$ denote the connected component of $p_n^{-1}(\mu)$ passing through the basepoint of $X^n_K$. Then $p_n$ restricts to a $k$-sheeted covering map $\mu_n \to \mu$ for some $1 \leq k \leq n$, and $\mu^k = (p_n)_*(\mu_n) \in \pi_n(K)$. As the image of $\mu$ generates the quotient $\pi(K)/\pi_n(K)$, it follows that $k = n$. Therefore, $\mu_n = (p_n)^{-1}(\mu)$ is a connected curve. As each component of $\partial X^n_K$ contains a lift of $\mu$, it follows that $\partial X^n_K$ is connected; and since the covering map restricts to an $n$-fold cyclic covering map $p_n : \partial X^n_K \to \partial X_K$, it follows that $\partial X^n_K$ is a torus. Also, since $\lambda \in \pi_\infty(K) \subset \pi^n(K)$, the map $p_n$ is one-to-one on the component $\lambda_n \subset p_n^{-1}(\lambda)$ passing through the basepoint of $\partial X^n_K$. In total, the pairs $(\mu_n, \lambda_n)$ and $(\mu, \lambda)$ give explicit coordinates
for us to identify \( \partial X^n_K \) and \( \partial X_K \) with \( S^1 \times S^1 \). With respect to these coordinates, we have shown that the covering map \( p_n \) on the boundary is simply given by the mapping \((x, y) \mapsto (x^n, y)\).

We glue a solid torus \( V^n \) to \( X^n_K \) along the slope \( \mu_n \) and a solid torus \( V \) to \( X_K \) along the slope \( \mu \). The construction of the first manifold is similar to Dehn surgery, except that now the space \( X^n_K \) is not presented as a knot complement but rather an arbitrary compact 3-manifold with torus boundary. The second manifold is, of course, just meridional filling along \( K \) not presented as a knot complement but rather an arbitrary compact 3-manifold with torus boundary.

A could imagine trying to prove this by establishing an isomorphism \( x \) a "subquotient" group of the knot group. Notice that if \( n \) is uniquely characterized by the properties just described. A mapping \( p \) covering on the complement, and is locally modeled in a neighborhood of \( p \), we can extend the covering map \( p_n \) across these solid tori via \( p_n : (x^n, y) \mapsto (x, y) \). We obtain a mapping \( p_n \cup p_n : X^n_K \cup V^n \to S^3 \) which maps \( p^{-1}(K) \) homeomorphically onto \( K \), is a cyclic \( n \)-to-1 covering on the complement, and is locally modeled in a neighborhood of \( p^{-1}(K) \) by the standard map \( p_n \). The resulting space is called the \( n \)-fold cyclic branched cover of \( K \) and denoted \( \Sigma_n(K) \). It is uniquely characterized by the properties just described.

Thanks to our expertise on how \( \pi_1 \) changes under filling, we see at once that

\[
\pi_1(\Sigma_n(K)) \cong \pi_1(X^n_K)/(\mu_n) \cong \pi^n(K)/(\mu^n),
\]

a “subquotient” group of the knot group. Notice that if \( x \in \pi(K) \), then \( x^n \) maps to 1 in the sequence (1), so it is contained in \( \pi^n(K) \). Consequently, we may consider its image in \( \pi_1(\Sigma_n(K)) \). Although \( \mu^n \) vanishes in this group, \( x^n \) in general will not.

As we shall see very soon, the cyclic branched coverings of a knot provide important information about it.

### 12.2 Colorings.

Recall that if \( D \) is a diagram for \( K \) and \( n \geq 1 \) is an odd positive integer, then we get an \( n \)-coloring of \( D \) by assigning each strand an element of the group \( C_n \). The assignment is subject to the restriction that if \( i, j, k \) denote the colors appearing on the strands incident with a crossing, then \( i + j \equiv k + k \) (mod \( n \)), where \( k \) denotes the color of the overstrand. We showed that the set \( A(D, n) \) of \( n \)-colorings of \( D \) are in one-to-one correspondence with the set \( \mathcal{R}(\pi(K), D_{2n}) \) of non-abelian representations \( \pi(K) \to D_{2n} \); the non-abelian restriction amounts to enforcing that a meridian maps to a reflection. Therefore, the number of \( n \)-colorings of \( D \) is an invariant of \( K \): it counts the number of these representations.

However, \( A(D, n) \) has further structure not immediately apparent from the representation description. Specifically, we can add and subtract two \( n \)-colorings to form a third. This makes \( A(D, n) \) into a finite abelian group. This group turns out to be an invariant of \( K \) and not just the diagram \( D \). You could imagine trying to prove this by establishing an isomorphism \( A(D, n) \cong A(D', n) \) whenever \( D \) and \( D' \) are related by a Reidemeister move. However, based on the philosophy that any invariant of a knot \( K \) is secretly an invariant of \( \pi(K) \), it is preferable to seek a natural description of this group in terms of \( \pi(K) \) itself.

To get started, notice that within \( A(D, n) \) sits the subgroup \( A_0(D, n) \) of \( n \)-colorings which take the value 0 on a distinguished strand in \( D \). We obtain a natural splitting \( A(D, n) \cong A_0(D, n) \oplus C_n \) by decomposing an arbitrary coloring into one in \( A_0 \) and a constant coloring. Therefore, focusing on
A_0 eliminates redundancy occurring in A. If \(\mu\) denotes the meridian surrounding the distinguished strand, then the correspondence between \(A(D, n)\) and \(\mathcal{R}(\pi(K), D_{2n})\) restricts to a correspondence between \(A_0(D, n)\) and representations \(\mathcal{R}_0(\pi(K), D_{2n})\) which send \(\mu\) to the fixed reflection \(r\) within \(D_{2n} = \langle \rho, r \mid \rho^n = r^2 = 1, r\rho r = \rho^{-1} \rangle\). Notice that any other reflection \(r'\) is conjugate to \(r\) by a unique rotation \(\rho^k\): the value \(k\) is characterized by the property that \(rr' = \rho^2k\) (note that we tacitly rely on the fact that \(n\) is odd). Thus, the action of \(C_n\) on \(D_{2n}\) by inner automorphisms gives rise to a decomposition \(\mathcal{R}(\pi(K), D_{2n}) \cong \mathcal{R}_0(\pi(K), D_{2n}) \times C_n\) (you might find it helpful to write out this correspondence).

In short, focusing on \(\mathcal{R}_0(\pi(K), D_{2n})\) eliminates redundancy occurring in \(\mathcal{R}(\pi(K), D_{2n})\), and we lose nothing by studying the correspondence between the smaller sets \(A_0(D, n)\) and \(\mathcal{R}_0(\pi(K), D_{2n})\).

We streamline the situation somewhat by collecting all the groups \(D_{2n}\) into their direct limit \(D\). Observe that any representation \(\pi(K) \to D\) has image equal to some \(D_{2n}\), since \(\pi(K)\) is finitely generated. Thus, forming the direct limit provides a framework for replacing all these various targets by one single group so that we need only study non-abelian representations \(\mathcal{R}(\pi(K), D)\). We may identify \(D\) with a subgroup of symmetries of the complex plane and \(r \in D\) with complex conjugation.

Within \(D\) sits \(C\), the direct limit of the finite cyclic groups \(C_n\). We may regard \(C\) as the group of rotations of the circle by a rational multiple of \(2\pi\): \(C = \{ \exp(2\pi it) \mid t \in \mathbb{Q} \} \subset \mathbb{C}^*\). We have \(D \cong C \times C_2\), where the generator \(r \in C_2\) acts by inversion: \(r x r^{-1} = x^{-1}\).

Now suppose that \(\varphi \in \mathcal{R}_0(\pi(K), D)\). The composite \(\pi(K) \to D \to ab(D) \cong C_2\) surjects, so it has kernel \(\pi^2(K)\), and \(\varphi\) restricts to a mapping \(\pi^2(K) \to C\). Additionally, \(\varphi([\mu]\overline{\mu}^2) = r^2 = 1 \in D\). Consequently, \(\varphi\) induces a map \(\pi^2(K)/(\overline{[\mu]}^2) \to C \subset \mathbb{C}^*\). Now we recognize this first group – it is the fundamental group of the branched double cover! Furthermore, since \(C\) is abelian, this map factors through \(H_1(\Sigma_2(K))\). In total, the mapping \(\varphi\) induces a unique character \(\chi : H_1(\Sigma_2(K)) \to C \subset \mathbb{C}^*\).

Therefore, colorings of a knot diagram give rise to characters on \(H_1(\Sigma_2(K))\); so already the branched double cover captures something interesting. As a challenge, see whether you can invert this correspondence: given a character \(\chi \in H_1(\Sigma_2(K))\) (with image contained in \(C\)), does it lift to a representation \(\varphi \in \mathcal{R}_0(\pi(K), D)\)? Is it easy to write down a putative \(\varphi\) in terms of \(\chi\)? If you have difficulty establishing that \(\varphi\) is an actual representation, where do you get stuck?