MT855: Combinatorial Methods in Knot Theory

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2 The Mazur Swindle

Wild knots help us probe the distinction between continuous and smooth topology in three dimensions, and they can even be used to tell us something interesting about smooth knots.

As a warm-up, consider the following amusing fact from algebra. Recall that if $R$ is a commutative ring and $P$ a module over $R$, then $P$ is called projective if it is a summand of a free $R$-module: $P \oplus Q \cong F$, where $Q$ and $F$ are $R$-modules and $F$ is free. Call $P$ strongly projective if we can take $Q$ to be free in this definition. The reason that you have never heard of strong projectivity is that it is equivalent to projectivity! Clearly strong projectivity implies projectivity. For the converse direction, suppose that $P \oplus Q \cong Q \oplus P \cong F$ exhibits $P$ as a projective $R$-module. Then

\[
P \oplus (F \oplus F \oplus \cdots) \cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots \\
\cong (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots \\
\cong F \oplus F \oplus \cdots
\]

exhibits $P$ as a strongly projective module! Of course, a more interesting question arises if we insist on finite rank in these definitions.

Now back to knot theory. A fundamental operation on knots is the connected sum operation, which is reminiscent of the direct sum operation we just used. Below is a picture definition of the connected sum of a pair of oriented knots $K$ and $J$.

![Figure 1: Knots $K$, $J$, and $K\#J$.](image)

It is instructive to consider why this operation is well-defined, but let’s accept this fact. By swinging the knot $J$ around clockwise, it is clear that $K\#J \simeq J\#K$. Alternatively, we can demonstrate this fact by shrinking $J$ down very small, moving it to the left through $K$, and reexpanding it to the left of $K$. This is a better perspective for showing that connected sum is a commutative operation for
multiple knots. The unknot is an identity for this operation, in that $K\# U \simeq U\# K \simeq K$ for every knot $K$.

Here is a simple but useful principle when performing isotopies of knots. Suppose that $K$ is a knot in $S^3$. Remove a small open ball surrounding a point on $K$. The result is a properly embedded arc $\kappa$ in the 3-ball $B^3$; that is, it is the image of a smooth embedding $I \hookrightarrow B^3$ such that $I \cap \partial B^3 = \partial I$ and $I \cap \partial B^3$. Two properly embedded, oriented arcs are isotopic if they are isotopic through properly embedded arcs. If two properly embedded, oriented arcs share the same pair of endpoints, then we may additionally request an isotopy rel endpoints. It is an exercise (modulo which foundational fact?) to show that isotopy rel endpoints gives the same notion of equivalence as ordinary isotopy for such arcs.

One may additionally consider other possible notions of equivalence: diffeomorphism (rel boundary) and smooth ambient isotopy (rel boundary). Check that they give the same equivalence relation on properly embedded arcs. Using any one, the principle is that $K \simeq K'$ as knots iff $\kappa \simeq \kappa'$ as arcs. In particular, if $K \simeq U$, then there exists an isotopy from $\kappa$ to a diameter of $B^3$.

The following result asserts that it is impossible to cancel a pair of non-trivial knots.

**Theorem 2.1** If $K\# J \simeq U$, then $K \simeq J \simeq U$. 

**Proof?** Suppose that $K\# J \simeq J\# K \simeq U$. Build a wild knot $W$ as pictured below, in the spirit of the infinite direct sum of projective modules from before.

![Figure 2: A wild knot W.](image)

Surround the first $K$-$J$ pair by a ball that meets $W$ in two points and avoids all the other knotting. In time $1/2$, cancel the first $K$-$J$ pair in this ball (i.e., isotop it to a horizontal line segment by the principle stated above). Similarly, in time $1/4$, cancel the second $K$-$J$ pair, in time $1/8$, cancel the third $K$-$J$ pair, etc. The end result is an isotopy from $W$ to the unknot. Now instead surround the first $J$ and the second $K$ by a ball. In time $1/2$, cancel this $J$-$K$ pair within its ball. Similarly, in time $1/4$, cancel the next $J$-$K$ pair, etc. The end result is an isotopy from $W$ to $K$. Therefore, $K \simeq U$. We can deduce that $J \simeq U$ in the same way by commuting each $K$, $J$ pair and repeating the argument with their roles switched. Alternatively, we just note that $U \simeq K\# J \simeq U\# J \simeq J$. 

This is Mazur’s swindle, and it ought to leave you wondering what exactly it proves. What happens when we cancel a $K$-$J$ pair in the above process? We may indeed conduct a smooth isotopy supported in each ball, but the composition of all these isotopies will typically not be a smooth map. So we just get a topological isotopy from $W$ to both of $U$ and $K$, so $K$ and $U$ are topologically isotopic. However, this is no surprise: any pair of smooth knots are topologically isotopic (we saw an example of this last
class, and you will prove it in generality on your homework). However, we can actually say a bit more: we may conduct a smooth ambient isotopy supported in each ball. Once again, the composition of all these isotopies will typically not be a smooth map, but what we get is a topological ambient isotopy between \( W \) and each of \( U \) and \( K \), so \( K \) and \( U \) are related by topological ambient isotopy. This is more significant, and it raises the following question:

**Question 2.2** Suppose that a pair of smooth knots are related by a topological ambient isotopy. Are they related by a smooth one?

Here is a sledgehammer proof that the answer is “yes.” If two smooth knots are related by a topological ambient isotopy, then they have homeomorphic complements, and the Gordon-Luecke theorem then shows that they are isotopic in the smooth category.\(^1\) I would like to see a more direct proof. If, as in the case of knot cancelation, one of the knots in the pair under consideration is the unknot, we get a somewhat lower-tech proof by applying the Dehn’s lemma plus the loop theorem. As we will see on a future homework, this result quickly leads to (and was motivated by) the fact that the unknot is determined by its complement (and its knot group).

Finally, we should mention that there are much lower-tech, emotionally comforting proofs of Theorem 2.1. For example, we will study the bridge number \( b(K) \) and Seifert genus \( g(K) \) of a knot. These invariants detect the unknot in the sense that if \( b(K) - 1 = 0 \) or if \( g(K) = 0 \), then \( K \simeq U \). (These facts are hidden on your current homework.) These invariants are also additive, in the sense that

\[ b(K \# J) - 1 = (b(K) - 1) + (b(J) - 1) \quad \text{and} \quad g(K \# J) = g(K) + g(J). \]

Either fact immediately establishes Theorem 2.1.

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\(^1\)Since we have not yet seen the proof of that theorem, one hopes that this is not circular reasoning. I’m pretty sure it isn’t.