MT855: Combinatorial Methods in Knot Theory

January 23, 2013

3 Homology and curves on surfaces, Part I

We begin with the calculation of the ordinary homology of a knot complement $S^3 \setminus K$.

Proposition 3.1 *If $K \subset S^3$ is a knot, then*

$$H_*(S^3 \setminus K) = \begin{cases} \mathbb{Z}, & * = 0, 1 \\ 0, & * \geq 2. \end{cases}$$

As remarked in Lecture 1, the proof is a basic application of the Mayer-Vietoris sequence, and it offers a helpful reminder of the maps involved involved in the sequence.

Proof. Let $\nu(K)$ denote an open tubular neighborhood of K. We apply the Mayer-Vietoris sequence to the decomposition $S^3 = \nu(K) \cup (S^3 \setminus K)$. Observe that the intersection $\nu(K) \cap (S^3 \setminus K)$ deform retracts onto the boundary of a slightly smaller regular neighborhood of K, which is homeomorphic to T^2, while $\nu(K)$ deform retracts onto its core, which is homeomorphic to S^1. Since $S^3 \setminus K$ is a connected, open 3-manifold, we get the desired statement at once for all $* \neq 1, 2$, so it suffices to examine the following portion of the Mayer-Vietoris sequence:

$$\begin{array}{c} H_3(\nu(K)) \oplus H_2(S^3 \setminus K) \longrightarrow H_3(S^3) \longrightarrow \\ \\ \longrightarrow H_2(T^2) \longrightarrow H_2(\nu(K)) \oplus H_2(S^3 \setminus K) \longrightarrow H_2(S^3) \longrightarrow \\ \\ \longrightarrow H_1(T^2) \longrightarrow H_1(\nu(K)) \oplus H_1(S^3 \setminus K) \longrightarrow H_1(S^3) \end{array}$$

which reduces to

$$0 \to \mathbb{Z} \to \mathbb{Z} \to H_2(S^3 \setminus K) \to 0 \to \mathbb{Z}^2 \to \mathbb{Z} \oplus H_1(S^3 \setminus K) \to 0.$$

We get the desired statement at once for $* = 1$, and we learn that $H_2(S^3 \setminus K)$ is a finite cyclic group. There are two ways to see that it is 0. The space $S^3 \setminus K$ deform retracts onto $X_K := S^3 \setminus \nu(K)$, and by Poincaré-Lefschetz duality and the Universal Coefficients Theorem we get

$$H_2(S^3 \setminus K) \cong H_2(X_K) \cong H^1(X_K, \partial X_K) \cong \mathrm{Hom}(H_1(X_K, \partial X_K), \mathbb{Z}).$$

Since the latter group is free abelian, it must vanish. Alternatively, recall that the map $H_k(S^3) \to H_{k-1}(T^2)$ is represented at the chain level by intersecting a k-cycle in S^3 with T^2 (assuming that a given class in $H_k(S^3)$ is represented by a smooth cycle transverse to T^2). When $k = 3$, it follows that the fundamental class of S^3 maps to that of T^2, so the map is an isomorphism and $H_2(S^3 \setminus K)$ vanishes by exactness. □

Notice the use of the tubular neighborhood theorem in this proof. What happens if we work instead in the topological category? For example, what is the homology of the complement of a wild knot like the one that appears in the Mazur swindle?

Back to the smooth setting, it will be preferable henceforth to switch our perspective from the open manifold $S^3 \setminus K$ to the compact manifold $X_K = S^3 \setminus \nu(K)$. We already used the deformation retraction from the first to the second in the above proof. Since $S^3 \setminus K$ is homeomorphic to the interior of X_K, it is clear that $S^3 \setminus K \cong S^3 \setminus K'$ if $X_K \cong X'_K$, and the converse implication is discussed in Gordon's Springer lecture notes.

The above proof teaches us a bit more. Let μ denote the homology class of $\partial D^2 \subset S^1 \times D^2 \cong \nu(K)$. A curve representing μ is a *meridian* for the knot K. Under the isomorphism

$$H_1(T^2) \xrightarrow{\sim} H_1(\nu(K)) \oplus H_1(X_K),$$

μ maps to 0 in the first factor, so its image (denoted by the same symbol) generates $H_1(S^3 \setminus K)$. The kernel of the map $H_1(T^2) \to H_1(X_K)$ is a summand of $H_1(T^2)$ generated by a unique (projective) primitive element $\pm\lambda$. Thus, a simple closed curve on ∂X_K representing the class λ is essential in ∂X_K and null-homologous in X_K. Such a curve is called a *Seifert longitude* for K. The homotopy type in ∂X_K of a meridian or Seifert longitude is uniquely determined modulo orientation. Here we are using the fact that $\pi_1(T^2) \cong H_1(T^2)$ in order to pass between homological and homotopical information. Are curves representing μ and λ unique up to *isotopy*? What about curves representing other homology classes? The following result tells all.

Theorem 3.2 (Homotopy-isotopy principle for curves; Baer (1927 or 1928)) *A pair of smooth, simple closed curves on a surface are homotopic iff they are isotopic.*

We will sketch a proof of this fact, closely following Farb and Margalit's *Introduction to Mapping Class Groups*, Chapter 1. It is fundamental, introduces useful techniques, and is just really cool, and proving it now will atone for my for not having learned it sooner.

To get started, consider the following basic problem: given a pair of smooth, simple closed curves $\alpha, \beta \subset S$, what is the minimum geometric intersection number between a pair of curves isotopic to α and β or, since we don't yet know that it's the same thing, homotopic to α and β? How can we recognize when a pair is in minimal position with respect to either isotopy or homotopy? It is not immediately obvious how difficult these problems are, but we will see that a simple principle governs their resolution!

Let us assume (without loss of generality) that $\alpha \pitchfork \beta$. One obvious indicator that α and β are *not* in minimal position (with respect to homotopy, a weaker requirement, and the one we will stick to when

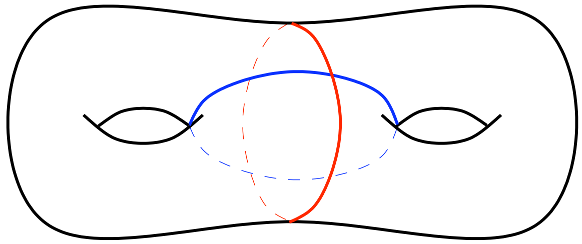

Figure 1: Curves in minimal position?

we use this term) is if there exists an embedded closed disk $D \subset S$ such that $int(D)$ is disjoint from α and β; ∂D contains two distinct points of intersection between α and β; and ∂D decomposes into two connected arcs, one contained in α and the other contained in β. In particular, the intersection points between α and β have opposite signs if we induce a local orientation on S supported nearby D. We call such a disk D a *bigon*. We may use D to guide an isotopy of α over β that removes this pair of intersections. Observe that we are using the 2-dimensional Schönflies theorem here, another foundational fact we should recognize when we apply. Also, notice that it is important that the two intersection points are distinct: compare the situation of a pair of cores of a Möbius strip that meet in a single intersection point.

Remarkably, this one obvious obstruction to being in minimal position is the only one.

Theorem 3.3 (Bigon criterion) *Two transverse simple closed curves $\alpha, \beta \subset S$ are in minimal position iff they do not cobound a bigon.*

Proof. Choose a pair of transverse simple closed curves $\alpha, \beta \subset S$ not in minimal position. We seek to exhibit a bigon between them. Let $H : S^1 \times I \to S$ denote a homotopy that reduces the intersection between α and β, so $\mathrm{im}(H(-,0)) = \alpha$ and $\mathrm{im}(H(-,1))$ is transverse to β and meets it in fewer points than α. We may assume that H is transverse to β. Therefore, the preimage $H^{-1}(\beta) \subset S^1 \times I$ consists of properly embedded arcs and simple closed curves. (As a sanity check, consider what happens for the case we handled with isotoping over a bigon, and in particular what it means for H to be transverse to β in this case.) By assumption on non-minimality, there exists at least one properly embedded arc δ with both endpoints on $S^1 \times \{0\}$. It cobounds a disk $D \subset S^1 \times I$ with an arc $\gamma \subset S^1 \times \{0\}$. At first, it seems like we can just use D to exhibit the desired bigon. Certainly $\gamma \cup \delta$ is null-homotopic, with null-homotopy guided by $H(D)$, and $\gamma \subset \alpha$, $\delta \subset \beta$. However, H might not embed D into S, so a little more work is needed. Stay tuned! . . .