4 Curves on surfaces, part II

Recall that we’re on course to prove the bigon criterion. Picking up from our last lecture, we hit a minor obstacle in locating a bigon during the proof. The relevant fix is contained in the following lemma.

**Lemma 4.1** Given two transverse simple closed curves $\alpha$, $\beta$ in a surface $S$, if some pair of lifts $\tilde{\alpha}$, $\tilde{\beta}$ to the universal cover $\tilde{S}$ intersect in more than one point, then $\alpha$, $\beta$ cobound a bigon.

**Proof of the bigon criterion (continued):** Let us assume Lemma 4.1 and finish the proof. Since $H(\gamma \cup \delta)$ is null-homotopic in $S$, it lifts to a closed curve in $\tilde{S}$, with $p^{-1}(\gamma)$ contained in a lift of $\alpha$ and $p^{-1}(\delta)$ contained in a lift of $\beta$. Therefore, these two lifts meet in more than one point, so by Lemma 4.1, $\alpha$ and $\beta$ cobound a bigon. $\square$

Now onto the proof of Lemma 4.1.

**Proof of Lemma 4.1:** The case $\chi(S) > 0$ is a simple exercise, so assume that $\chi(S) \leq 0$, so that the universal cover $\tilde{S}$ is homeomorphic to $\mathbb{R}^2$ or $I \times \mathbb{R}$. Let $p$ denote the covering map. Suppose that $\tilde{\alpha}$ and $\tilde{\beta}$ meet in two or more points. Then there exist subarcs of $\tilde{\alpha}$ and $\tilde{\beta}$ that cobound a disk $D_0 \subset \mathbb{R}^2$ (Schönflies). Consider $p^{-1}(\alpha) \cap p^{-1}(\beta) \cap D_0$. By compactness and transversality, there exists an *innermost* embedded disk $D \subset D_0$, i.e., such that $\partial D$ consists of one arc of $p^{-1}(\alpha)$ and one arc of $p^{-1}(\beta)$ and $\text{int}(D)$ is disjoint from $p^{-1}(\alpha) \cup p^{-1}(\beta)$.

We claim that $p$ embeds $D$ in $S$, so that $p(D)$ produces the required bigon. Let $v_1, v_2 \in \partial D$ denote the two intersection points, $\tilde{\alpha}_1$, $\tilde{\beta}_1 \subset \partial D$ the arcs between them, and $x_i = p(v_i)$, $i = 1, 2$. We argue that $p$ is injective on $\partial D$. First, $x_1 \neq x_2$, since $v_1$ and $v_2$ have opposite signs of intersection. Second, no point in $\text{int}(\tilde{\alpha}_1)$ maps to $x_1$ or $x_2$, since otherwise $p^{-1}(\beta)$ would meet $\text{int}(\tilde{\alpha}_1)$, contradicting the inner-most property. Third, no two points of $\text{int}(\tilde{\alpha}_1)$ map to the same point under $p$, for otherwise there would be a point of $\text{int}(\tilde{\alpha}_1)$ that maps to $x_1$, which we just argued does not occur. Thus, $p$ is injective on $\tilde{\alpha}_1$, and by symmetry, it is injective on $\tilde{\beta}_1$. Fourth, $p(\text{int}(\tilde{\alpha}_1)) \cap p(\text{int}(\tilde{\beta}_1)) = \emptyset$, again using the inner-most property. Thus, $p$ is injective on $\partial D$.

Now suppose that $p(x) = p(y)$ for arbitrary $x, y \in D$. Then $\phi(x) = y$ for some deck transformation $\phi$. Observe that $\phi(D)$ is an embedded disk with $\partial(\phi(D)) = \partial(\phi(D))$. Suppose that $\phi(\partial D) \cap \partial D = \emptyset$. Then $\phi(D)$ is contained in $\text{int}(D)$ or $\mathbb{R}^2 \setminus D$ by the Jordan curve theorem. However, the first case
cannot occur, since $\phi(\partial D)$ contains a portion of $p^{-1}(\alpha) \cup p^{-1}(\beta)$, while $\text{int}(D)$ is disjoint from it. The second cannot either, since then $y = \phi(x) \in \phi(D) \subset \mathbb{R}^2 \setminus D \subset \mathbb{R}^2 \setminus \{y\}$, which we wrote out for style points. Therefore, $\phi(\partial D) \cap \partial D \neq \emptyset$. However, since $p$ is injective on $\partial D$, it follows that $\phi$ is the identity map, so $x = y$. It follows that $p$ is injective, and $p(D)$ is the required bigon. \hfill \Box

The above proof made effective use of an “innermost” argument. We will see arguments of this sort frequently in the course.

In class, we tried to base our proof of the bigon criterion on the fact that a simple closed curve in a surface is null-homotopic iff it bounds an embedded disk. A proof of this fact goes essentially like that of Lemma 4.1. One takes the curve $\epsilon$ and lifts it to a simple closed curve $\tilde{\epsilon} \subset \mathbb{R}^2$ that bounds a disk $D$. One difference is that while we could argue above that $\text{int}(D)$ does not meet $\tilde{\alpha} \cup \tilde{\beta}$, we do not readily have the analogous fact that $\text{int}(D)$ does not contain $\phi(\tilde{\epsilon})$. The “fix” in this case is that if it did, then $\phi(D) \subset \text{int}(D)$, and Brouwer’s fixed point theorem implies that $\phi$ has a fixed point; but $\phi$ is a deck transformation, so it must be the identity map.\footnote{Ian Biringer pointed out that you can deduce that a null-homotopic curve in a surface bounds a disk by an application of van Kampen’s theorem and the classification of surfaces. Of course, the latter also involves some amount of effort (triangulability of surfaces or somesuch).}

In class we tried to apply this to $H(\gamma \cup \delta)$. We wanted to say that by choosing $\delta$ innermost, we got that $H$ embeds $\gamma \cup \delta$ into $S$, and that $H(D)$ is a null-homotopy. However, I think there is an issue about making $H$ one-to-one on $\delta$: the map $H$ could make $\delta$ wrap around $\beta$ several times. One could further examine what happens inside $S^1 \times I$ in this case, but I think it is just as well to argue as we did above, following Farb-Margalit.

With the bigon criterion in hand, we now sketch a proof of the homotopy-isotopy principle for curves.

**Proof sketch of the homotopy-isotopy principle for curves:** We just handle the case of $S = T^2$, which is the principal case of interest to us. For other surfaces, consult Farb-Margalit. Thus, suppose that $\alpha$ and $\beta$ are homotopic simple closed curves on $T^2$. Isotop $\beta$ to meet $\alpha$ transversely. If the two curves intersect, then they are not in minimal position, since $\beta$ is homotopic to a parallel push-off of $\alpha$ which does not intersect $\alpha$ at all. Thus, if they intersect, then the bigon criterion guarantees that they cobound a bigon $D$. Use the bigon to guide an isotopy of $\beta$ over $D$ so as to lower the number of intersection points between $\beta$ and $\alpha$. Continue to do so until having isotoped $\beta$ disjoint from $\alpha$, referring to it by the same name. Now there are two possibilities. If $\beta$ is non-separating, then $\alpha$ is a null-homologous simple closed curve in $T^2 \setminus \nu(K) \cong S^1 \times I$. Therefore, $\alpha$ and $\beta$ cobound an annulus (are we using some foundational fact about plane topology here?), and we use the annulus to complete the isotopy from $\beta$ to $\alpha$. If instead $\beta$ is separating, then it bounds a disk $D \subset T^2$. We use $D$ to isotop $\beta$ to a round circle of radius $\epsilon$, having put an arbitrary flat metric on $T^2$ (again, what are we using about the plane?). Since $\beta$ separates, the same is true of $\alpha$, so we isotop it to a round circle of radius $\epsilon$ as well. Now these two round circles are isotopic by a translation, and that completes the isotopy between $\beta$ and $\alpha$. \hfill \Box
Returning to our motivating question, the homotopy-isotopy principle provides us with a clean understanding of isotopy classes of simple closed curves on $T^2$. Recall that a primitive element $x$ in a free abelian group $F$ is one that is not a non-trivial multiple of any other class: if $x = ny$, $n \in \mathbb{Z}$, $y \in F$, then $n = \pm 1$. An exercise is to show that $x$ is primitive iff it extends to a basis of $F$.

**Proposition 4.2** A non-zero homology class in $H_1(T^2)$ is represented by a simple closed curve iff it is primitive. If $\mu, \lambda$ is a basis of $H_1(T^2)$, then a class $p\mu + q\lambda$ is primitive iff $\gcd(p, q) = 1$.

We will turn to the proof next class and make great use of it.