6 Dehn surgery, continued

We have a shortage of knots at the moment. To address this, let \( U \subset S^3 \) denote the unknot. Then the curve of slope \( p/q \) on \( \partial X_U \subset S^3 \) gives a knot in \( S^3 \). By definition, this is the \((p,q)\)-torus knot \( T(p,q) \). For example, with \( p/q = 3/2 \), we get the right-hand trefoil knot, and with \( p/q = -3/2 \) we get the left-hand trefoil knot. Notice that if one of \( p \) or \( q \) is \( \pm 1 \), then the resulting knot is just the unknot.

Your third homework explores properties of these knots from the point of view of the knot group.

More generally, if \( K \subset S^3 \) is a non-trivial knot, then the curve of slope \( p/q \) gives a knot \( C_{p,q}(K) \), which is called a \(|q|\)-cable of \( K \). Notice that if \( q = \pm 1 \), then the resulting knot is isotopic to \( K \). If \( p = 1 \) and \( q \neq 0 \), then we get the unknot. Now, however, we could get something interesting if \( p = 1 \) and \( q \neq 0 \). The cabling operation is an instance of the more general satellite operation, which we will take up when we discuss the coarse classification of knots according to the geometry of their complements.

Meanwhile, we were supposed to calculate the homology of \( K(p/q) \):

**Proposition 6.1** The first homology group of \( K(p/q) \) is isomorphic to \( \mathbb{Z}/p\mathbb{Z} \). The second homology group is isomorphic to 0 if \( p \neq 0 \) and \( \mathbb{Z} \) if \( p = 0 \).

As you might expect, the proof is another application of the Mayer-Vietoris sequence.

**Proof.** Decompose \( K(p/q) \) into \( X_K \) and \( V \) to apply the Mayer-Vietoris sequence, where \( V \) denotes the glued-in solid torus. (Technically, we should thicken one of \( X_K \) or \( V \) slightly in order to apply the sequence, but it will not matter). As before, \( V \) has the homotopy type of \( S^3 \) and \( V \cap X_K \cong \partial X_K \) is a torus. Now the simplicity of \( H_*(X_K) \) will pay dividends. We focus on \( H_1(K(p/q)) \). The relevant portion of the sequence reads

\[
H_1(\partial X_K) \to H_1(V) \oplus H_1(X_K) \to H_1(K(p/q)) \to 0.
\]

We justify the last 0 by using the Mayer-Vietoris sequence for reduced homology, noting that \( \Pi_0(\partial X_K) \) vanishes, since \( \partial X_K \) is connected. Let \( \mu_V \) denote the class of a meridian in \( \partial X_K \). Then there exists a curve in \( \partial X_K \) representing a homology class \( \lambda_V \) such that \( \mu_V, \lambda_V \) forms a basis for \( H_1(\partial X_K) \) and \( \lambda_V \) maps onto a generator of \( H_1(V) \), which we denote by the same name. Let \( \mu, \lambda \in H_1(\partial X_K) \) denote the standard basis for the knot complement. Thus, \( \mu_V = p\mu + q\lambda \). It follows that with respect to the bases \((\mu_V, \lambda_V)\) for \( H_1(\partial X_K) \) and \((\lambda_V, \mu)\) for \( H_1(V) \oplus H_1(X_K) \), the first map in the above sequence
is expressed by the matrix \( \begin{pmatrix} 0 & p \\ 1 & * \end{pmatrix} \). This matrix has cokernel isomorphic to \( \mathbb{Z}/p\mathbb{Z} \), which establishes the first assertion of the Proposition. By Poincaré duality, \( H_2(K(p/q)) \cong H^1(K(p/q)) \), and by the Universal Coefficients Theorem, this is isomorphic to the free part of \( H_1(K(p/q)) \). This establishes the second assertion of the Proposition.

We already mentioned some interesting consequences of this Proposition last class. One thing you see right away is that \( K(1/n) \) is a \( \mathbb{Z}HS^3 \) for any knot \( K \) and \( n \in \mathbb{Z} \). Your second homework reveals the easy fact that \( K(p/q) \) is a \( \mathbb{Q}HS^3 \) as long as \( p/q \neq 0 \). In time we’ll see that we get lots of distinct manifolds by varying \( K \) and \( p/q \). As a preview, Dehn showed that 1-surgery along the right hand trefoil produces the Poincaré homology sphere, which we investigated last class. There is a great paper by Kirby and Scharlemann, “Eight faces of the Poincaré homology sphere,” which explores various constructions for this space and how they are equivalent. When \( K \) equals the unknot we get a family of spaces called lens spaces. These will be central examples in this course, and we’ll discuss them more thoroughly soon.

Now we address the fact that \( K(p/q) \) is determined up to homeomorphism by the notation.

**Proposition 6.2** The homeomorphism type of \( K(p/q) \) depends only on the terms involved in the notation: the knot \( K \subset S^3 \) and the slope \( p/q \).

We will prove this through the following sequence of steps.

**Proposition 6.3 (Isotopic gluings give homeomorphic spaces)** Suppose that \( M \) and \( N \) are compact manifolds and \( \varphi : \partial M \to \partial N \) a homeomorphism of their boundaries. Then the homeomorphism type of the space \( M \cup_\varphi N \) depends only on the isotopy type of \( \varphi \).

The basic idea behind the proof of this result is to break the two manifolds into \( N \), a collar neighborhood of \( \partial M \) in \( M \), and the rest of \( M \). Then use the isotopy to define a homeomorphism between the collar neighborhoods occurring in the two decompositions, and extend it by the identity map to obtain a homeomorphism between the two manifolds. Your third homework asks you to think through this more carefully.

However, this result does not quite put us into position to establish Proposition 6.2, since there exist non-isotopic maps from \( \partial V \) to \( \partial X_K \) which both send \( \mu_V \) onto a curve of slope \( p/q \). (Can you see this?)

**Proposition 6.4 (The homotopy-isotopy principle for surface homeomorphisms; Baer (1927-28))**

Two homeomorphisms between connected, compact surfaces are homotopic iff they are isotopic, with two exceptions: the (isotopy classes of the) identity map and reflection in the disk, and the identity map and inversion in the annulus.

Baer established this result using the homotopy-isotopy principle for curves in surfaces. The main case of interest is that of the torus. Isotopy classes of homeomorphisms of a surface \( S \) constitute its
mapping class group \text{Mod}(S)$. The group operation is composition. The orientation-preserving homeomorphisms form a subgroup \text{Mod}^+(S). I recommend Rolfsen’s book for a treatment of Proposition 6.4 for the case of the torus, as well as the next fact.

**Proposition 6.5 (The mapping class group of the torus)** \text{Mod}(T^2) \cong GL_2(\mathbb{Z}) and \text{Mod}^+(T^2) \cong SL_2(\mathbb{Z}).

**Proof of Proposition 6.5:** By Proposition 6.4, it suffices to identify the group of orientation-preserving homeomorphisms of $T^2$ modulo homotopy. By homotopy theory, homotopy classes of maps \( \varphi \in [X, K(G, 1)] \) are parametrized by their induced maps \( \varphi_* \in \text{Hom}(\pi_1(X), G) \). (What is the issue with basepoints here?) Taking \( G = \mathbb{Z}^2 \), we may take \( T^2 \) as both \( X \) and a representative of \( K(\mathbb{Z}^2, 1) \). Therefore, \( [T^2, T^2] \cong \text{Hom}(\mathbb{Z}^2, \mathbb{Z}^2) \). Taking \( \varphi \) a homeomorphism implies that \( \varphi_* \) is an isomorphism, so \( \text{Mod}(T^2) \subset [T^2, T^2] \) maps isomorphically onto a subgroup of \( \text{Isom}(\mathbb{Z}^2) = GL_2(\mathbb{Z}) \). To show that is surjects, let \( A \in GL_2(\mathbb{Z}) \). Then \( A \) is an automorphism of \( \mathbb{Z}^2 \), and it extends to an automorphism of \( \mathbb{R}^2 = \mathbb{Z}^2 \oplus \mathbb{R} \). This extension descends in turn to a homeomorphism of \( T^2 = \mathbb{R}^2/\mathbb{Z}^2 \) whose induced map on \( \pi_1(T^2) \) is the given map \( A \). Therefore, we get the first statement in the Proposition. For the second statement, simply note that an orientation-preserving homeomorphism of \( T^2 \) induces a class in \( SL_2(\mathbb{Z}) \). \( \square \)

Propositions 6.3, 6.4, and 6.5 still do not put us in position to establish Proposition 6.2 for the reason already given. In fact, we’ve gotten a little off course with Proposition 6.5, but this suite of result controls what happens when we glue two manifolds with torus boundary, as we shall elaborate next class. What we need to get to Proposition 6.2 is the Alexander trick. This will let us decompose \( V \) into the “puck” neighborhood \( I \times D_V \) of a meridian disk and its complement, glue the two pieces in one at a time, and apply Propositions 6.3 and the Alexander trick to define the homeomorphism between \( X_K \cup_\varphi V \) and \( X_K \cup_\psi V \), where \( \varphi \) and \( \psi \) take \( \mu_V \) to curves of slope \( p/q \). We wrap up this line of argument next class.

Class also featured discussion of an amusement. An easy mnemonic for remembering the Euler characteristic of the circle and line segment is that they are their own Euler characteristics. A challenge is to find other topological spaces that are their own Euler characteristics. More meaningfully, for every \( n \in \mathbb{Z} \), write down an expression of \( n \) whose Euler characteristic is \( n \). For example, to get 2, we can form \( 4 - 2 \) (if you, like TeX and Robert, close the tops of your 4’s) or \( \sqrt{16} \) (if you, like Lisa and perversions of TeX, do not).