Speaking of crossings, suppose that there are 17 ghosts and 17 ghostbusters in the plane. Each ghostbuster targets a ghost with its pack, but ghostbusters’ beams cannot cross. Is it possible for each ghostbuster to target a different ghost simultaneously? Clearly not, if, for instance, they all fall in a line with ghosts to one side and ghostbusters to the other. Let us suppose, then, that the 34 creatures are in general position, meaning that no three fall on a line. In this case, it is possible. First, pair the ghosts and ghostbusters arbitrarily and have the ghostbusters shoot their targets. If there exists a pair of crossing beams, then have two of the ghostbusters involved switch targets. The triangle inequality implies that the total length of beams decreases. Therefore, if we pair the ghosts and ghostbusters so as to minimize the total beam length over all 17! possible pairings, then no pair of beams will cross, and we have the desired massacre.

A question arose during the lecture: given an initial pairing, what is the maximum number of beam switches required to get down to a crossingless matching? For example, is it a polynomial in \( n \)?

The result generalizes:

**Theorem 0.1** (Akiyama-Alon (1989)). Given \( d \) sets of \( n \) points in \( \mathbb{R}^d \), with all \( dn \) points in general position, there exist \( n \) pairwise disjoint simplices such that each simplex has one vertex from each set.

“General position” means that no \( d + 1 \) points lie in a \( d \)-plane. The proof we gave in the case \( d = 2 \) does not appear to generalize. I have not thought seriously about this, but I do not know a sensible auxiliary function, like the sum of areas, and a version of the triangle inequality that would work. Instead, the proof follows a divide-and-conquer method, which we will see again in our work on the Szemerédi-Trotter theorem and the Erdős distance problem.

**Proof.** We proceed by induction on \( n \). When \( n = 1 \), the result is immediately true. Suppose then that the result is true for sets of fewer than \( n \) points. Let \( S_1, \ldots, S_d \) denote the collection of \( d \) sets of size \( (n - 1)/2 \) in each of these two

We have had to pause the proof because we did not yet state this result. Here it is:

**Theorem 0.2** (Point Partitioning Lemma). If \( S_1, \ldots, S_d \) are finite sets in \( \mathbb{R}^d \), then there exists a \( (d - 1) \)-plane \( H^{d-1} \) that simultaneously bisects each: that is, the number of points of \( S_j \) in each half-space determined by \( H^{d-1} \) is equal, for \( j = 1, \ldots, d \).

(You get to choose whether we mean “open” or “closed” half-space.)

With that in hand, we now resume:

\( \triangleright \) If \( n \) is odd, then there is at least one point of each \( S_i \) on \( H \). Since the points are in general position, it follows that there is exactly one point of each \( S_i \) on \( H \). Make a simplex on these points. Each open half-space of \( H \) contains \( (n - 1)/2 \) points of each \( S_i \). By induction, we can make pairwise disjoint rainbow simplices on the \( d \) sets of size \( (n - 1)/2 \) in each of these two
half-spaces. Together with the simplex on $H$, we get the desired rainbow simplices. If instead $n$ is even, $H$ may contain an even number of points of the various $S_i$, but still no more than $d$ points total. A little geometric argument shows that we can tilt $H$ so as to obtain a new bisecting hyperplane $H'$ that does not meet any $S_i$ at all. An induction as before closes this case and hence the proof.

To keep the lecture self-contained, we prove the Point Partitioning Lemma:

Proof: Let $\epsilon > 0$, and let $A_i(\epsilon)$ denote an $\epsilon$-neighborhood about each $S_i$. There exists a bisecting $(d-1)$-plane $H^d_{\epsilon^{-1}}$ for these sets, by the Ham Sandwich theorem. □

We have had to pause the proof again to state what this is:

**Theorem 0.3** (Ham Sandwich Theorem). If $A_1,\ldots,A_d$ are measurable subsets of $\mathbb{R}^d$, then there exists a $(d-1)$-plane $H^d_{\epsilon^{-1}}$ that simultaneously bisects all $d$: that is, the measure of $A_j$ intersected with each half-space determined by $H^d_{\epsilon^{-1}}$ is half of the measure of $A_j$, for $j = 1,\ldots,d$.

The case $d = 1$ is clear, but for $d = 2$ this is already a non-trivial result due to Banach. Its generalization to $d \geq 3$ is due to Stone (of Erdős-Stone fame) and Tukey.

Let us resume:

$\triangleright$ Each half-plane $H_\epsilon$ has the description $H_\epsilon = \{x \in \mathbb{R}^d \mid x \cdot v_\epsilon = a_\epsilon\}$ for a unit vector $v \in \mathbb{R}^d$ and a non-negative real number $a$. Since $\bigcup A_i(\epsilon)$ is bounded in $\epsilon$, it follows that the values $a_\epsilon$ are, too. Therefore, there exists an accumulation point $(v,a)$ for the values $(v_\epsilon,a_\epsilon)$ as $\epsilon \to 0$. The resulting half-space $H = \{x \in \mathbb{R}^d \mid x \cdot v = a\}$ does the job. □

Now to keep the lecture self-contained, we prove the Ham Sandwich Theorem:

Proof. Embed $\mathbb{R}^d$ as the level set $\{x_{d+1} = 1\} \subset \mathbb{R}^{d+1}$. It is tangent to the top of the unit sphere $S^d \subset \mathbb{R}^{d+1}$. For each point $x \in S^d$, take the $d$-plane $H(x) \subset \mathbb{R}^{d+1}$ orthogonal to $x$ and through the origin. The point $x$ points in the direction of one of the half spaces $S(x)$ cut off by $H(x)$. Let $f(x) = (\mu(A_1 \cap S(x)), \ldots, \mu(A_d \cap S(x)))$: it is the $d$-tuple of measures of the intersections of the sets $A_i$ with the half-space $S(x)$. A little argument in real analysis shows that $f$ is a continuous function, at least if we assume that each $A_i$ has finite measure (if not, we can still get by with a little care). The pair $\pm x$ guaranteed by the Borsuk-Ulam theorem for this $f$ gives the half-space $H(x) = H(-x)$ that simultaneously bisects each of the $A_i$. □

Once more we have to state and prove the quoted auxiliary result:

**Theorem 0.4** (Borsuk-Ulam). If $f : S^n \to \mathbb{R}^n$ is a continuous function, then there exists a pair of antipodal points $\pm x \in S^n$ for which $f(x) = f(-x)$.

This result is very versatile, as have already glimpsed. It expresses a sort of invariance of domain. It has many equivalent formulations, two more of which appear below, and another of which you will play around with on problem set. We will give an intuitive argument lifted from Matoušek’s book, but our proof has a small gap in it I would like someone to fill in.

**Proof.** First, we state an equivalent formulation: if $g : S^n \to \mathbb{R}^n$ is a continuous function with the property that $g(-x) = -g(x)$ for all $x \in S^n$, then $g$ has a zero. To see that they are
equivalent, first apply the stated theorem to $g$: the function takes some pair of antipodes to the same point, but it always takes antipodes to negatives of one another, so it must take these antipodes to 0. Conversely, given any $f$, form the function $g(x) = f(x) - f(-x)$ and apply the reformulation to get the desired result.

Next, suppose that we have a counterexample to the reformulation. We will show that there is a smooth counterexample to it, too. By compactness, a counterexample $g$ satisfies $|g(x)| \geq \epsilon > 0$ for some $\epsilon$ and all $x \in S^n$. By smooth approximation, there exists a smooth function $g_1$ that is $(\epsilon/3)$-close to $g$: $|g_1(x) - g(x)| < \epsilon/3$ for all $x$. The function $g_2(x) = g_1(x) - g_1(-x)$ is smooth, satisfies $g_2(-x) = -g_2(x)$, and is $(2\epsilon/3)$-close to $g$, so $|g_2(x)| > \epsilon/3 > 0$ for all $x$. Therefore, $g_2$ is the desired smooth counterexample.

Now we argue that there is no smooth counterexample. For this, consider the projection map $p: S^n \to \mathbb{R}^n$ onto the first $n$ coordinates. This function satisfies $p(-x) = -p(x)$ for all $x$, and it takes the values 0 at just two (antipodal) points, which we label $N$ and $S$. Take the straight-line homotopy $H: S^n \times [0, 1] \to \mathbb{R}^n$ from $g_2$ to $p$: $H(x, t) = tg_2(x) + (1-t)p(x)$. This $H$ is a smooth homotopy. Consider $H^{-1}(0)$. By the hypothesis on $g_2$, $H^{-1}(0)$ is disjoint from the top of the cylinder $S^n \times \{1\}$. Now, if 0 were a regular value of $H$, then $H^{-1}(0)$ is a smooth, properly embedded 1-dimensional submanifold of the cylinder. It also has an antipodality, i.e. a free $\mathbb{Z}/2\mathbb{Z}$ mapping $(x, t) \to (-x, t)$. Consider the component of it containing $N \times \{0\}$. Its other endpoint cannot be $S \times \{0\}$, since otherwise we would get an antipodality on a compact interval, which cannot occur. Therefore, its other endpoint must be on $S^n \times \{1\}$, giving a 0 of $g_2$, a contradiction. \hfill $\square$

The big “if” is whether 0 is a regular value of $H$. You may need to adjust $g$ or $H$ a little to make it work. If you see a nice way of doing so, please let me know!

For completeness, here is a nice non-intuitive argument.

**Second proof.** Let us suppose the theorem is false. Take a counterexample $f$. Define

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}.$$ 

This defines a map $g: S^n \to S^{n-1} \subset \mathbb{R}^n$ with the property that $g(-x) = -g(x)$ for all $x \in S^n$. As such, it descends to a mapping on projective spaces $\bar{g}: \mathbb{R}P^n \to \mathbb{R}P^{n-1}$. Such maps exist (for instance, the constant map), but this map has a further feature: an arc between a pair of antipodes maps to an arc between a pair of antipodes. It follows that, once we choose basepoints, $\bar{g}$ induces an isomorphism on fundamental groups:

$$\bar{g}_*: \pi_1(\mathbb{R}P^n) \to \pi_1(\mathbb{R}P^{n-1}).$$

Both of these groups are $\mathbb{Z}/2\mathbb{Z}$ for $n \geq 2$. As a result, it follows that

$$\bar{g}^*: H^1(\mathbb{R}P^{n-1}; \mathbb{Z}/2\mathbb{Z}) \to H^1(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$$

is an isomorphism: this follows, since $H^1(\mathbb{R}P^k; \mathbb{Z}/2\mathbb{Z}) \approx \text{Hom}(\pi_1(\mathbb{R}P^k), \mathbb{Z}/2\mathbb{Z})$. The cohomology ring $H^*(\mathbb{R}P^k; \mathbb{Z}/2\mathbb{Z})$ is isomorphic as a graded ring to $\mathbb{Z}/2\mathbb{Z}[\alpha_k]/(\alpha_k^{k+1})$, where $\alpha_k$ denotes the generator in degree 1. Since $\bar{g}^*(\alpha_{n-1}) = \alpha_n$, and $\bar{g}^*$ defines a ring homomorphism, we
obtain

\[ 0 = \overline{g}^*(\alpha_{n-1}^n) = \overline{g}^* (\alpha_{n-1})^n = \alpha_n^n \neq 0, \]

a contradiction. \[\square\]