We will resume with neighborhood complexes in a moment; first, an amusing interlude.

A matriarch wishes to bequeath her golden necklace with eight rubies and six emeralds to her two daughters, who have much smaller necklines than she does. She takes the necklace to a jeweler and requests that the necklace be cut and rearranged into two smaller necklaces, each with the same number of the two types of gemstone. Naturally, she wishes the number of cuts to be minimized, so as to ensure the integrity of the smaller necklaces (less soldering, etc.). How many cuts are required to form the smaller necklaces?

If the eight rubies occur all in a row, followed by the six emeralds, then two cuts are required, and two suffice: cut the necklace between the middle two rubies and between the middle two emeralds; then join the two end pieces into one, and the two smaller necklaces are formed. On your homework, you have the chance to argue directly that two cuts are sufficient, regardless of the distribution of the rubies and emeralds, and furthermore with the numbers eight and six replaced by any pair of positive even numbers.

Another matriarch in a very similar predicament brings a very similar necklace to the jeweler, except that her necklace additionally features ten diamonds. How many cuts are required to form the smaller necklaces in this case? Now three cuts may be required, if, for instance, diamonds precede rubies precede emeralds in sequence. Do three cuts suffice?

The following general result answers this and similar questions:

**Theorem 0.1** (Necklace Partitioning Theorem). Given a necklace with $d$ different types of gemstones, with an even number of each type, $d$ cuts suffice to equipartition the gemstones into two groups.

Putting the gemstones in series as above shows that $d$ cuts may be necessary. Remarkably, all known proofs of the necklace partitioning theorem are topological.

**Proof.** The jeweler pays a visit to his brother’s $(d+1)$-dimensional delicatessen next door. He carefully lays the necklace out along a subset of the image of the moment curve $\gamma : \mathbb{R} \to \mathbb{R}^d$, $\gamma(t) = (t, t^2, \ldots, t^{d+1})$. The moment curve has the remarkable property that any $d$-dimensional affine hyperplane meets it in at most $d$ points. For suppose that $\gamma(t_1), \ldots, \gamma(t_{d+1})$ lie on a $d$-dimensional affine hyperplane. As in the proof of the Ham Sandwich Theorem, we position $\mathbb{R}^d$ as the $\{x_1 = 1\}$ hyperplane in $\mathbb{R}^{d+1}$. A $d$-dimensional hyperplane in $\mathbb{R}^d$ extends to a $d$-dimensional linear hyperplane in $\mathbb{R}^{d+1}$ (i.e. a hyperplane containing the origin). Since $\gamma(t_1), \ldots, \gamma(t_{d+1})$ lie in this hyperplane, it follows that they are linearly dependent. However, a linear dependence implies that the Vandermonde determinant

$$\begin{vmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^d \\ 1 & t_2 & t_2^2 & \cdots & t_2^d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_{d+1} & t_{d+1}^2 & \cdots & t_{d+1}^d \end{vmatrix}$$
vanishes. This determinant has the value $\prod_{j<k}(t_j - t_k)$, as you can check on the homework. With distinct values $t_1, \ldots, t_{d+1}$, this determinant is non-zero. Hence the remarkable property of the moment curve is established. Now apply the Point Partitioning Lemma of Lecture 10 to the set of gemstones, so arranged in $\mathbb{R}^d$. There exists an affine hyperplane that simultaneously bisects each set of gemstones, and this hyperplane meets the necklace in no more than $d$ points. Hence it gives rise to the desired cut.

Now we resume our treatment of Lovász’s proof of Kneser’s conjecture.

First, some background on simplicial complexes and topological spaces. A (finite) simplicial complex $K$ consists of a vertex set $V$ and a family of subsets of $V$ that is closed under taking subsets. Formally, $K$ is this family of subsets, and $V$ is a separate set, the ground set. Thus, $K$ could be thought of as the edges of a special kind of hypergraph on $V$. The elements of $K$ are called the simplices of $K$. The significance of simplicial complexes has to do with their topology. Given a simplicial complex, it has a geometric realization $\overline{K}$. Since we only deal with finite simplicial complexes, we can construct $\overline{K}$ concretely, as follows. We identify the points of $V$ with an orthonormal basis of $\mathbb{R}^n$, $n = |V|$. The convex hull of $\sigma \in K$ is a subset $\sigma \subset \mathbb{R}^n$. Their union is $K$. It is topologized as a subspace of $\mathbb{R}^n$.

Connectivity. A topological space is $c$-connected if every map of a $d$-dimensional sphere into it can be extended to a map of a $(d+1)$-dimensional ball, for all $d = 0, 1, \ldots, c$. Equivalently, the homotopy groups $\pi_d$ vanish for all $d = 0, \ldots, c$. (Strictly speaking, $\pi_0$ is only a set; we just mean that it has cardinality one.)

A simplicial map $f : K_1 \to K_2$ of simplicial complexes consists of a map of the underlying ground sets $f : V_1 \to V_2$ that takes simplices of $K_1$ to simplices of $K_2$. A simplicial map determines a continuous map $\overline{f} : \overline{K}_1 \to \overline{K}_2$. The map is given by $f$ on the vertex set and extends to the uniquely defined affine map over each geometric simplex.

Here, again, are the components in Lovász’s proof of Kneser’s conjecture:

**Theorem 0.2.** If $\overline{N}(G)$ is $(k-1)$-connected, then $G$ is not $(k+1)$-colorable.

**Theorem 0.3.** Let $S$ be a finite set and $n, k$ natural numbers. Let $K$ be the simplicial complex whose vertices are the $n$-subsets of $S$ and whose simplices are collections of $n$-subsets $A_0, \ldots, A_m$ such that

$$|\bigcup_{i=0}^m A_i| \leq n + k.$$ 

Then $\overline{K}$ is $(k-1)$-connected.

For $|S| = 2n + k$, $K$ is the neighborhood complex of $KG_{n,k}$. Since its realization is $(k-1)$-connected, $KG_{n,k}$ is not $(k+1)$-colorable. Hence its chromatic number is $k + 2$.

Given a simplicial complex $K$, it has a barycentric subdivision $sdK$. The ground set of $sdK$ is the set of simplices of $K$, i.e. $K$ itself. The simplices of $sdK$ are chains $\sigma_1 \subset \cdots \subset \sigma_d$ of simplices of $K$. We can think of forming $sdK$ by placing a point down in the centroid of the geometric simplex $\overline{\sigma}$. These points constitute its ground set. We get a simplex in
Taking a collection of centroids that span a geometric simplex contained in $\overline{K}$. The construction makes it clear that $sdK \approx K$.

The advantage to working with $sdN(G)$ over $N(G)$ in this setting is that it comes with a natural simplicial mapping $C N$ to itself that very much resembles the antipodal map. More precisely, a vertex $A \in sdN(G)$ is a subset of $V$ with a common neighbor, and we define $C N(A)$ to be the set of its common neighbors, another vertex of $sdN(G)$. This is a simplicial map of $sdN(G)$: a sequence of subsets $A_1 \subset \cdots \subset A_m$ with a common neighbor maps to a sequence of subsets $CN(A_1) \supset \cdots \supset CN(A_m)$ with a common neighbor. This simplicial map has no fixed points: $CN(A) \neq A$ for all $A$, for the simple reason that no vertex neighbors itself. Furthermore, the induced map $CN : sdN(G) \to sdN(G)$ has no fixed points, either. (Contrast with the simplicial map from a 1-simplex to itself that exchanges its two endpoints.) For if it fixes some point $x$, then it maps the unique simplex containing $x$ in its interior onto itself. In this case, the simplex takes the form $A_1 \subset \cdots \subset A_m$, and its image under $CN$ takes the form $CN(A_1) \supset \cdots \supset CN(A_m)$. For the first to map onto the latter, it follows that $A_1 \supset CN(A_m)$; but $A_1 \subset A_m$, so vertices of $A_1$ neighbor themselves, a contradiction. A more distinctive property is:

\[ CN^3 = CN, \quad \overline{CN}^3 = \overline{CN}. \]

A picture clearly illustrates why the first is the case (the second follows formally), and moreover why $CN^2 \neq Id$ in general. Thus, $\overline{CN}$ is nearly an antipodality (i.e. a free $\mathbb{Z}/2\mathbb{Z}$ action) on $\overline{N(G)}$. In fact, the two properties show that it is an antipodality on im($\overline{CN}$) $\subset N(G)$.

(Note: what follows did not actually appear in the lecture; instead, we began to prove Theorem 0.2, which we write out in the next set of lecture notes.)

It is worth contemplating im($\overline{CN}$) (and its realization im($\overline{CN}$)) for a moment. Its vertices are subsets of $V$ of the form $CN(A)$, $A \subset V$. Its simplices are, as before, chains of subsets of this form under inclusion. From a few pictures, you begin to believe that im($\overline{CN}$) is a deformation retraction of $\overline{sdN(G)}$. Actually, it is clear what the retraction map should be: $\overline{CN}^2$. We just need to show that it is homotopic to the identity map on $sdN(G)$.

We do so by means of a pleasant and more general method to recognize homotopic maps:

**Lemma 0.4 (Homotopic Maps Lemma).** If $f$ and $g$ are continuous maps from a space $X$ to the realization of a simplicial complex $K$, and for every $x \in X$ there exists a simplex $\sigma \in K$ such that $[\sigma]$ contains both $f(x)$ and $g(x)$, then $f$ and $g$ are homotopic.

The hypothesis of the lemma holds if, for instance, $f$ and $g$ are induced by simplicial maps between simplicial complexes $K_1$ and $K_2$, and for every simplex $\sigma_1 \in K_1$, there exists a simplex $\sigma_2 \in K_2$ containing the images of $\sigma_1$ under both maps.

**Proof.** The homotopy is a “straight-line” homotopy: let $f(x)$ move to $g(x)$ by unit speed inside $[\sigma]$. I leave to you the details that this mapping is well-defined (easy) and continuous (also not so bad).

Let’s recapitulate:
Proposition 0.5. The subspace $\text{im}(CN)$ is a deformation retraction of $\text{sdN}(G)$. The geometric realization of the map $CN^2 : sdN(G) \to \text{im}(CN)$ is the retraction mapping.

Proof. We will apply the Homotopic Maps Lemma to the identity map and $CN^2$. Given a simplex $A_1 \subset \cdots \subset A_m$ in $sdN(G)$, its image under $CN^2$ is $CN^2(A_1) \subset \cdots \subset CN^2(A_m)$. Each subset of $V$ in the last sentence is a subset of $CN^2(A_m)$. That is to say, these are all vertices in the subdivision of $CN^2(A_m)$ in $sdN(G)$. Therefore, these two simplices are contained in a common simplex, and the Homotopic Maps Lemma applies. □