We begin this lecture with a description of de Grey's construction of a finite unit distance graph $N$ in the plane with chromatic number 5. We proceed through de Grey's arXiv preprint.

To begin, let $H$ denote the 7-vertex unit distance graph induced on the vertices of a regular hexagon of side length 1, together with its center. This graph $H$ admits many 4-colorings. In some of these, one color class has size 3. That is too big. We call a 4-colored copy of $H$ slim if it has no color class of size 3; moreover, we call a 4-colored unit distance graph slim if every copy of $H$ within it is slim. We will produce two finite unit-distance graphs, one called $L$ and one called $M$, that both contain copies of $H$. The graph $L$ contains 52 copies of $H$, and we will show that no 4-coloring of $L$ is slim. The graph $M$ contains a distinguished copy of $H$, and through the use of a computer, de Grey shows that in any 4-coloring of $M$, this distinguished copy of $H$ is not slim. We then draw a copy of $L$ on the plane and print off 52 copies of $M$ with its highlighted copy of $H$ onto transparencies. We put these transparencies in a stack on top of $L$, so that the $j$-th copy of $M$ has its highlighted copy of $H$ sitting on top of the $j$-th copy of $H$ inside $L$. If we attempt to 4-color $N$, then we get 4-colorings of $L$ and each copy of $M$. However, one of the copies of $H$ inside of $L$, say the $j$-th one, must have three vertices all the same color, while the distinguished copy of $H$ in the $j$-th copy of $M$ cannot have three vertices all the same color; since these are the same copy of $H$ inside of $N$, we get a contradiction. Therefore, this graph $N$ therefore cannot be 4-colored at all!

We begin by describing $L$, which is smaller than $M$ and whose asserted property can be checked by hand. Take the tiling of $\mathbb{R}^2$ by regular hexagons of side length 1 (similar to what we used to 7-color $\mathbb{R}^2$, except there the side length there was chosen differently). Take one of these hexagons and the six that surround it. The graph $J$ is the unit distance graph induced on the vertices of these hexagons and their centers. How many hexagons does $J$ contain? This question resembles a brain-teaser. Observe that $J$ contains a central copy of $H$, the six copies of $H$ that we used to surround $H$, and six more whose centers are 1 away from its center, for a grand total of 13. There is also a hexagon of side length two we call $2H$. This $J$ is $1/4$ of the way to our $L$. It admits several slim colorings, which can be usefully enumerated and analyzed. As de Grey shows, up to symmetry (of $J$, of which there are $|D_6|=12$, and of the colors, of which there are 4!), and ignoring the colors on the 12 vertices outside of $2H$ (i.e. furthest from the center), there are just 6 slim colorings. I refer you to his paper for them, and we saw some in lecture. In these colorings, the 12 maximally distant vertices can be colored in many ways so that the outer copies of $H$ are slim. Note as well that there are other 4-colorings of $J$ in which the inner 7 copies of $H$ are slim, but they cannot be completed to a slim 4-coloring of $J$, i.e. in which all of the outer copies of $H$ are slim, too. We drew one in the lecture.

Let us examine the vertices of $2H$ in the slim colorings of $J$. In two of the six colorings, they all get the same color as the center of $J$. In another two of the six colorings, four consecutive get the same color as the center of $J$, and the other two get one other color. In the last two of the six colorings, two opposite get the same color as the center of $J$, and the other four get one other color. Now take a fixed copy of $J$ and rotate it about its center until each vertex
of $2H$ rotates to a point 1 away from where it started. The union of $J$ and its rotant $J'$ is a new graph $K$. This is similar to how we constructed the Moser spindle. This $K$ contains $2 \times 13 = 26$ copies of $H$, and it is halfway to the desired $L$. Consider a slim 4-coloring of $K$. It must induce slim 4-colorings of $J$ and $J'$. Based on our analysis of the slim colorings of these (isomorphic) graphs, the induced 4-colorings of $J$ and $J'$ must both be of the third type. For instance, you can think about why the coloring of $J$ cannot be of the second type and the coloring of $J'$ of the third type. Thus, in a slim 4-coloring of $K$, every pair of opposite vertices of $2H$ in $J$ have the same color. Now fix one vertex $A$ of $2H \subset J \subset K$ and rotate $K$ about it until the opposite vertex $B$ of $2H$ moves to a position $C$ at distance 1 from where it began. The union of $K$ and its rotant $K'$ is the required graph $L$ with 52 copies of $H$. For suppose that $L$ had a slim 4-coloring. It induces a slim 4-coloring of both of $K$ and $K'$, but then $B$ and $C$ are forced to be the same color as $A$, which is prohibited, since $B$ and $C$ are adjacent. Hence $L$ is the desired graph with no slim 4-coloring.

The construction of $M$ is about as simple as the construction of $L$, although it is larger, and it takes a computer to verify its desired property. As de Grey writes, the inspiration behind constructing $M$ was to get a graph with many Moser spindles arranged quite densely together that would somehow force a distinguished copy of $H$ to have three vertices all the same color in any 4-coloring. We only sketch its construction. We place a number of Moser spindles with tips at a single vertex. We get a graph $V$ in which the central vertex has degree 30 and whose edges fall into translates of 5 rotated copies of an equilateral triangular lattice. We take a copy of $V$ and translate it to a copy centered at each vertex of $V$, giving a graph $W$. We then take a translate of (a trimmed down copy of) $W$ centered at each vertex of a copy of $H$. The result turns out to play the role of the graph $M$: in any 4-coloring of $M$, the copy of $H$ is not slim.

de Grey discusses both how he checked that $M$ has the stated property, as well of ways of reducing the size of its (and $N$’s) construction. Perhaps this construction will inspire one of you to give a construction of a stand-in for $M$ whose required property can be checked without the use of a computer...?

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We now change gears and address the final topic of the semester: the polynomial method. While not quite as recent as de Grey’s work, it is a very exciting fairly recent development (from the past decade) that continues to have wide ramifications throughout combinatorics. You need look no further than Larry Guth’s excellent book on the topic, and we will just dip into the beginning of it. Not only is the mathematics in Guth’s book fascinating, but he provides a lot of enlightening commentary, as well.

The method is mysteriously strong. Its strength is evocative of the probabilistic method.

To begin, we will prove two results of Zeev Dvir, which initiated this method. The first is the finite field Nikodym problem. It is so-named because of a construction of a paradoxical set by Nikodym. Consider a subset $S$ of the unit square in $\mathbb{R}^2$ with the property that for every point $x \in S$, there exists a line $L$ passing through $x$ that meets $[0, 1]^2$ precisely in $S$. It is not hard to come up with examples of such subsets: after we looked at a couple of examples based on lines, Chris suggested that we could take $S$ to be the graph of any function from $[0, 1]$ to
[0, 1]. In this case, the certificate lines \( L \) can all be chosen vertical. What Nikodym did was very surprising: he gave a construction of such a set \( S \) that has full measure, i.e. area 1. This set \( S \) is therefore paradoxical: it is large, in the sense that it fills up the square in terms of its area, but it is still small, in the sense that for every target \( x \in S \), you can shoot an arrow through the square that flies through and just hits \( S \) in \( x \).

We follow custom by defining a Nikodym set to be a subset \( N \subset [0, 1]^2 \) with the property that for all \( x \in [0, 1]^2 - N \), there exists a line \( L \) containing \( x \) and such that \( L - \{x\} \) is contained in \( N \). Thus, \( N \) is the complement to a set \( S \) as above. It is therefore large in the sense of containing lots of line segments, while Nikodym gives an example of a Nikodym set that is small, in the sense of having measure 0.

Thus, Lebesgue measure is not the best way to express the largeness imposed by the defining condition on \( N \). Hausdorff dimension instead seems to be a better measure, and we will say a little bit more about this in connection with Kakeya sets in the next lecture. (Aside: Nelson, of Hadwiger-Nelson fame, administered my French exam in graduate school. He selected a passage from Bourbaki on point set topology for me to read, and all went well until I mistranslated “separable” to mean “separable.” “No,” he said, “that’s not what it means. Think of some separation axioms that you know.” “I don’t know,” I said, “Hausdorff?” “That’s right,” he replied. Ha!) Showing that Nikodym and Kakeya sets are large in the sense of Hausdorff dimension is a major problem in (Fourier) analysis.

Tom Wolff proposed in the early ’90’s to look at finite analogues of these problems to get a sense for their difficulty and whether they could shed any light on the original problems. In this setting, we define a Nikodym set to be a subset \( N \subset \mathbb{F}_q^n \) with the property that for all \( x \in \mathbb{F}_q^n - N \), there exists a line \( L \) containing \( x \) with the property that more than half (i.e. \( \geq q/2 \)) of the points of \( L - x \) lie in \( N \). (Note that this is a little less restrictive than the notion we introduced above, but it will only serve to indicate the strength of the results that we will obtain.) Must \( N \) be large? Note that \( |N| \geq q/2 \) is trivial. What should we mean by “large”? A sensible notion is that \( |N| \) should be a large fraction of \( |\mathbb{F}_q^n| = q^n \). If we fix a dimension to work in (like \( n = 2 \)), then we might hope for a fraction that does not depend at all on the order of the input field. Indeed, this is the case:

**Theorem 0.1** (Dvir (2009)). A Nikodym set \( N \subset \mathbb{F}_q^n \) has size \( |N| \geq c_n q^n \), for a constant \( c_n \) independent of \( q \). In fact, we may take \( c_n = (10n)^{-n} \).

The proof of this result and the follow-up, the finite field Kakeya problem, requires the use of an auxiliary polynomial. Its existence is non-constructive, similar to our use of the probabilistic method. The idea is to argue that if \( N \) were small, then we could produce a non-zero polynomial \( f \) of quite low degree vanishing on \( N \). The reason is that polynomials are abundant; the existence of \( f \) is guaranteed by parameter counting. Then the defining feature of \( N \) will show that \( f \) actually has to vanish on all of \( \mathbb{F}_q^n \). However, polynomials are resilient: a polynomial of too low degree that vanishes on all of \( \mathbb{F}_q^n \) must be the 0 polynomial. This is a contradiction! Therefore, \( N \) must be rather large.

We will see the details in the next lecture!