Recall two results from last class:

**Theorem 0.1** (Turán’s theorem (1941)). For all \( n, t \geq 1 \), \( \text{ex}(n, K_{t+1}) = |E(G(n, t))| \), where \( G(n, t) \) denotes the complete \( t \)-partite graph on \( n \) vertices with parts of nearly equal size. Moreover, \( G(n, t) \) is the unique extremal example.

It is a little exercise to sort out from this result that

\[
\text{ex}(n, K_{t+1}) = \left( 1 - \frac{1}{t} \right) \frac{n^2}{2} - \frac{r(t-r)}{2t},
\]

where \( r \) denotes the least positive residue of \( n \) (mod \( t \)). A useful way to view this result is to look at the multiplier on \( n^2/2 \) as an edge density and to see the part that does not contain \( n \) as negligible. If the edge density of a graph exceeds \( 1 - 1/t \), then it contains a copy of \( K_{t+1} \).

**Theorem 0.2** (Erdős-Stone (1946), Erdős-Simonovits (1966)). Fix a graph \( H \) with an edge. For all \( \epsilon > 0 \), there exists \( n_0 \) such that for all \( n \geq n_0 \),

\[
\text{ex}(n, K) \leq \text{ex}(n, H) \leq \text{ex}(n, K) + \epsilon n^2,
\]

where \( K \) denote the complete graph on \( \chi(H) \) vertices.

Bollobás calls this result the fundamental theorem of extremal graph theory. We will not prove it, but I encourage you to look up the proof and discuss. A proof appears in Lovász’s Combinatorial Problems and Exercises, 10.38.

The lower bound is clear, in view of Turán’s theorem, since for \( t = \chi(H) - 1 \), \( G(n, t) \) realizes the lower bound and does not contain a copy of \( H \), since it is \( t \)-colorable. The result is saying that edge density in excess of \( (1 - 1/t) \) will force a copy of \( H \), once we look at graphs with sufficiently many vertices relative to the excess. That is, you cannot beat Turán’s bound by much.

Problem: give an example of a graph \( H \) with \( \chi(H) > 2 \) and for which \( \lim_{n \to \infty} \text{ex}(n, H) - \text{ex}(n, K) = \infty \).

When \( \chi(H) = 2 \), the Erdős-Stone theorem does not tell us much: for any positive edge density \( \epsilon > 0 \), a graph with sufficiently many vertices and edge density \( \epsilon \) will contain a copy of \( H \). The case of bipartite graphs closely relates to the Zarankiewicz problem from 1951. Zarankiewicz effectively asked for the value \( \text{ex}(n, K_{s,t}) \). We will consider a more precise version of the question on homework and again later in the course.

The first interesting case to consider is \( H = C_4 \). Reiman gave an interesting construction in this case to show that \( \text{ex}(n, C_4) \) is somewhat large. It is our first glimpse of the interaction between graph theory and algebra / geometry. Let \( \mathbb{F} \) denote a finite field of \( q \) elements. The affine plane \( \mathbb{F}^2 \) contains \( q^2 \) points. It also contains \( (q+1)q \) lines: each line has one of \( q+1 \) slopes in \( \mathbb{F} \cup \{1/0\} \), and the lines of a chosen slope partition \( \mathbb{F}^2 \) into \( q \) parts of size \( q \). Each pair of points is contained in a unique line, and each pair of lines intersect in at most one point. We make an **incidence graph** \( G_q = (V, E) \) whose vertex set is bipartite, consisting
of the points and lines of $\mathbb{F}^2$, and we put in an edge between a point and a line iff that point lies on that line. This graph $G_q$ has $\sim q^2$ points, $\sim q^3$ edges, and no $C_4$. Therefore, there are examples of graphs with $n$ vertices, no $C_4$, and $\sim n^{3/2}$ edges. Here we are using the notation $f \lesssim g$ for real-valued functions $f$ and $g$ to mean that there exists a constant $C > 0$ so that $f \leq C \cdot g$, and $f \sim g$ to mean that $f \lesssim g$ and $g \lesssim f$. In other words, $f \sim g$ if the ratio $f/g$ stays bounded between two positive values.

Remarkably, Reiman’s construction gives the correct bound. Establishing an upper bound on $\text{ex}(n, K_{s,t})$ is not much harder than the special case $s = t = 2$, and it introduces the use of Hölder’s inequality, so we proceed straight to it.

**Theorem 0.3** (Kővari-Sós-Turán (1957)). $\text{ex}(n; K_{s,t}) \lesssim s^{1/t} n^{2-1/t} + tn$.

If $s$ is fixed, then the first term dominates when $n \gg t$, while the second term dominates otherwise. This kind of dichotomy of behavior will recur, as in the Szemerédi-Trotter theorem.

**Proof.** Suppose that $G$ is a $K_{s,t}$-free graph on $n$ vertices. Count the number of copies of $K_{1,t}$ in $G$. On one hand, it equals

$$\sum_{v \in V} \binom{\deg(v)}{t},$$

by counting how many $K_{1,t}$ contain a given $v$ as the star vertex. On the other hand, it is at most $(s-1)\binom{n}{t}$, by counting how many $K_{1,t}$ may contain a given $t$-subset of $V$, and using the condition that $G$ is $K_{s,t}$-free. Note that for the special case $s = t = 2$, we are just counting two-edge paths and saying that no two two-edge paths have the same pair of endpoints in a $C_4$-free graph. Hence

(1) $$\sum_{v \in V} \binom{\deg(v)}{t} \leq (s-1)\binom{n}{t}.$$ 

Now we must estimate both sides of this bound. We will use the soft estimates $|x - t|^t/t! \leq t(x/t)^t$, with the idea that with $t$ fixed and $n$ large, both $n$ and most vertex degrees will be large, so there is not much loss. However, some vertex degrees could be $< t$, and then the lower bound is not valid, since $(\frac{t}{t})$ vanishes in this domain. To cope with this, we simply let $V' \subset V$ be the set of vertices with degree $\geq t$, and then we get

$$\sum_{v \in V'} (\deg(v) - t)^t \leq (s-1)n'^t.$$ 

We would like to emulate what we did in Mantel’s theorem, applying Cauchy-Schwarz. Now we need Hölder’s inequality (or Jensen’s inequality). In an analysis book, this would read $||fg||_1 \leq ||f||_p ||g||_q$, where $\frac{1}{p} + \frac{1}{q} = 1$. For sequences $(a_i), (b_i)$, it reads $\sum |a_i b_i| \leq (\sum |a_i|^p)^{1/p}(\sum |b_i|^q)^{1/q}$.

We apply it to the sequences $(a_i) = (\deg(v) - t)$ and $(b_i) = (1)$, where $v$ ranges over $V'$. We obtain

$$\sum_{v \in V'} (\deg(v) - t)^t \leq \left( \sum_{v \in V'} (\deg(v) - t)^t \right)^{1/t} n^{1-1/t}.$$
Adding back terms $\deg(v) - t$ to the left side with $v \in V - V'$ does increase it, so we add them and apply the Handshake theorem to get $2|E| - nt$ as a lower bound. Rearranging,

$$(2|E| - nt)^t/n^{t-1} \leq \sum_{v \in V'} (\deg(v) - t)^t.$$ 

Comparing back to (1) gives

$$\frac{(2|E| - nt)^t}{n^{t-1}} \leq (s-1)n^t.$$ 

Rearranging gives the stated bound.

How sharp is the exponent in the KST estimate? Reiman’s construction shows that it is sharp for $K_{2,2}$. It is useful to consider what happened to make that work out. For point-line incidences in the projective plane, every pair of lines determines a point, and every pair of points determines a line. Therefore, about half of the pairs of vertices in the incidence graph are endpoints of a $K_{1,2}$, so the left and right sides of (1) are equal, up to a factor of two. All vertex degrees are large, so our soft estimates on the binomial coefficients are good. Furthermore, all of the vertex degrees are the same, so the estimate using Hölder’s inequality is tight.

Brown (~ 1960) gave a construction of $K_{3,3}$-free graphs on $n$ vertices with $\sim n^{5/3}$ edges. Hence $ex(n, K_{3,3}) \sim n^{5/3}$. His construction is based on looking at point-sphere incidences in $\mathbb{F}_q^3$. One of the problems (in Guth’s book) is to go through this construction in detail. It does not appear to generalize.

The best bounds we have on $ex(n, K_{t,t})$ for $t = 4$ come from Brown’s construction and KST. For $t \geq 5$ they are $n^{2-2/(t+1)} \lesssim_{t} ex(n, K_{t,t}) \lesssim_{t} n^{2-1/t} + tn$, where $\lesssim_{t}$ means up to a function of $t$ alone. The lower bound comes from a random construction, which we will see in a couple of weeks. Note that it is no better than Brown’s construction for $t = 5$. Sharpening these bounds is a major open problem.

Here is a wonderful application of KST back to geometry.

Let $S$ be a set of $n$ points in the plane. How many pairs of these points lie at a unit distance apart? For $n = 3$, all $\binom{3}{2}$ may, if the points form the vertices of an equilateral triangle of side length 1. For $n = 4$, as many as 5 may, but not 6: no 4 points lie at pairwise distance one apart. By looking at points in a grid of size $\sqrt{n} \times \sqrt{n}$, Erdős showed that as many as $n^{1+c/\log \log n}$ pairs of points can determine a unit distance. He has offered $500 if you can beat this or prove that it cannot be beaten.

In the way of upper bounds, we can at least show that $< 2/3$ of the pairs can form a unit distance. Form a graph $G(S)$ with vertex set $S$ and edge set the pairs of points that lie at a unit distance apart. By the remark we already made, $G(S)$ is $K_4$-free. Therefore, Turán’s theorem applies and delivers the stated bound.

We can do better by seeing that $G(S)$ avoids a bipartite configuration. Two unit circles meet in at most two points. This means that two points in $S$ are at pairwise distance one
from at most two other points in $S$. Therefore, $G(S)$ is $K_{3,2}$-free. By Kővari-Sós-Turán, it follows that $G(S)$ contains $\lesssim n^{3/2}$ edges.

Later on, we will get a bound of $\lesssim n^{4/3}$. This is the state of the art! In dimension three, the state of affairs is somewhat less embarrassing. See the problem set.