Imagine we have \( s \) kilns in a brickyard and \( t \) shipping depots. We wish to make paths between the kilns and the depots so as to minimize the number of crossings between pairs of paths. What is the fewest number of pairs of crossing paths that we can arrange? This is Turán’s famous brickyard problem, which apparently occurred to him while working at an internment camp during World War II. (Chris Ratigan raised a different and interesting problem that we will not explore, which is to permit multiple paths to meet in a single crossing and to minimize the number of such crossings, disregarding multiplicity. This is somewhat similar to point-line incidence problems to which we will return.)

The standard drawing of \( K_{s,t} \) in the plane contains \( \binom{s}{2} \cdot \binom{t}{2} \) crossings: that is because for each pair of the \( s \) vertices and each pair of the \( t \) vertices, they determine a unique pair of edges that cross, and conversely, each crossing determines the pair of endpoints of two edges that cross at it. A slightly more clever drawing of \( K_{s,t} \) realizes \( \left\lfloor \frac{s}{2} \right\rfloor \cdot \left\lfloor \frac{s-1}{2} \right\rfloor \cdot \left\lfloor \frac{t}{2} \right\rfloor \cdot \left\lfloor \frac{t-1}{2} \right\rfloor \) crossings, and the assertion that you can do no better is the content of a famous conjecture.

For \( s = t \), this construction gives \( \sim t^4 \) crossings. Is this the correct order of magnitude? As we shall see, it is. The proof proceeds by two applications of the so-called amplification technique, as I learned about from a very informative (as usual) blog post due to Terry Tao that I recommend you check out. The proof feels quite magical and it pertains to any graph; previously, I never understood what makes it tick, but I think that Tao does a nice job of contextualizing it, at least to the point that I have an instinct as to how I could try to mimic the approach in other settings.

As a warm-up, let’s treat graph drawing a bit more generally. A graph \( G = (V,E) \) is planar if it can be drawn in the plane without edge crossings. Most of us are familiar with a famous result characterizing planar graphs due to Kuratowski, and sometimes attributed to Pontryagin as well. To state it, we make an informal definition: a graph \( H \) is a subdivision (or topological minor) of a graph \( G \) if you can obtain \( H \) from \( G \) by replacing each edge in \( G \) by a path. This is somewhat more restrictive than saying that \( H \) is simply a minor of \( G \), which would mean that it can be obtained from \( G \) by deleting and contracting edges and deleting isolated vertices.

**Theorem 0.1** (Kuratowski (1930’s)). A graph \( G \) is planar if and only if it does not contain a subdivision of \( K_5 \) or \( K_{3,3} \).

We will not prove this theorem, although it would probably just take one lecture to do so. There are many variations and amplifications of this result. One often sees the version of this theorem with the weaker conclusion that \( G \) contains \( K_5 \) or \( K_{3,3} \) as a minor, from which the stated version is not hard to derive (you may do this for homework). In the latter form, Kuratowski’s theorem is the prototype of a forbidden minor characterization, of which there are many very beautiful results (to do with drawing graphs on other surfaces; to do with embedding graphs into \( \mathbb{R}^3 \) so that cycle(s) do not link or knot; and including the graph minor

\[1\text{Phrasing due to Dan Archdeacon.}\]
theorem of Robertson and Seymour, perhaps the deepest result in graph theory.) In a future lecture, we will address a variation on this theorem due to van Kampen and Flores.

Suppose that \( G = (V, E) \) is a planar graph, and consider a planar drawing of it. (The phrasing is admittedly a bit clumsy; Marius Huber relayed an amusing observation by Lukas Lewark, who delightedly pointed out that English lacks a good counterpart to the German distinction between admitting a planar drawing [“eben” / “plän...”] and being drawn in the plane [“planar”, a tragic coincidence], analogous to how a manifold could be orientable vs. oriented.) A face is, to be precise, the closure of a component of the complement of the planar drawing of \( G \). Assuming that \(|V| \geq 3\), we may add edges to \( G \) to get a supergraph \( G' = (V, E') \) and a planar drawing thereof in which every face is homeomorphic to a disk with no vertices or edges in its interior. Let \( F \) denote the set of faces in this drawing. Thus, each edge in \( G' \) is in boundary of exactly two faces in \( F \). Thus, by the “faceshake” theorem, the total number of edges around each face, added over all faces, is \( 2|E| \). Since \( G \) (and \( G' \)) are assumed to be simple, the minimum number of edges around a face is 3. Therefore, \( 3|F| \leq 2|E'| \). By Euler’s formula, \(|V| - |E'| + |F| = 2\). Combining the two, we obtain \(|E| \leq |E'| \leq 3|V| - 6\).

Using this bound shows that \( K_5 \) is non-planar. However, it does not quite work to show that \( K_{3,3} \) is non-planar. We can easily remedy this shortcoming. Note that for a bipartite graph \( G \), the same method above gives \(|E| \leq 2|V| - 4\), since every face is surrounded by at least 4 sides. This bound shows that \( K_{3,3} \) is non-planar.

The main take-away at this point is this: if \( G = (V, E) \) is a graph, \(|V| \geq 3\), and and \(|E| - 3|V| > 0\), then any drawing of \( G \) in the plane contains a crossing. (We could take “−6” in place of “0”, but this rough estimate will serve our purposes.) Let \( cr(G) \) denote the minimum number of crossings in an drawing of \( G \) in the plane. Thus, we have

\[
|E| - 3|V| > 0 \implies cr(G) > 0.
\]

We can “amplify” this in the following way, vaguely reminiscent of the alteration method lower bound on Ramsey numbers.

**Proposition 0.2.** \( cr(G) \geq |E| - 3|V| \).

**Proof.** Induct on \(|E| - 3|V| \). The result holds in the base case that this value equals 1. When it larger than 1, take a drawing of \( G \) in the plane, and locate a crossing. Remove one edge that forms the crossing. The result is a drawing of a subgraph with one fewer edge, so it contains at least \(|E| - 3|V| - 1\) crossings by induction, which together with the one we removed yields the desired bound. \( \square \)

In fact, we could redefine \( cr(G) \) to denote the minimum number of crossings between edges with disjoint endpoints. For homework, you can show that Proposition 0.2 still holds with this somewhat more restrictive definition.

Now we come to a ridiculous result, the crossing number lemma, the first result that we have seen that was proven during (one of) our lifetimes:

**Theorem 0.3** (Ajtai-Chvátal-Newborn-Szemerédi (1982), Leighton (1983)). If \( G \) is a graph with \(|E| > 4|V| \), then any drawing of \( G \) in the plane contains at least \(|E|^3/(64|V|^2)\) crossings (between edges with disjoint endpoints).
Theorem 0.3. Let $G$ be a graph with $c$ vertices and $e$ edges, and let $K_n$ be a complete graph on $n$ vertices. Then the number of complete subgraphs of $G$ with at least $k$ vertices is at most $c^k e^{k/2}$.

Proof. Take a drawing of $G = (V, E)$ in the plane, and say that it has $c$ crossings between edges with disjoint endpoints. The idea is that we will “amplify” the bound in Proposition 0.2 by applying it to a suitably chosen subgraph of $G$ that still reflects the global structure of $G$ well enough to conclude something about $G$. We do so by selecting a subgraph of $H = (V', E')$ at random, by choosing each vertex independently with probability $p$ apiece, and taking precisely the edges that connect the chosen vertices. A given edge of $G$ therefore appears in $H$ with probability $p^2$, and a given crossing of $G$ appears with probability $p^4$. Let $X_v$, $X_e$, and $X_c$ denote the random variables that count the number of vertices and edges in this randomly constructed subgraph $H$ and the number of crossings in the induced drawing of it. Thus, we have $E(X_v) = p|V|$, $E(X_e) = p^2|E|$, $E(X_c) = p^4c$. On the other hand, the number of crossings in $H = (V', E')$ is larger than $|E'| - 3|V'|$. That is, $X_c - X_v + 3X_e > 0$, so the same applies to its expectation: $E(X_c) - E(X_v) > 0$. By linearity of expectation, it follows that $p^4c - p^2|E| + 3p|V| > 0$. Rearranging, $c > p^{-2}|E| - 3p^{-3}|V|$. This bound holds for any value $p \in (0, 1)$. We have been in this position before. Now choose $p$ to maximize the right-hand side in terms of $|V|$ and $|E|$. You can do this with calculus, but a short-cut is to choose $p$ so that the two terms involved on the right side are the same function of $|V|$ and $|E|$, up to a constant multiplier (this reminds me of dimensional reasoning from physics). Taking $p = 4|V|/|E|$ - which is less than one, by assumption! - leads to the stated bound on $c$. \hfill $\square$

The proof should make you pause to reflect quite a lot. Why is $p \sim |V|/|E|$ the right value for building a random subgraph? What sorts of variations on this theme are there? I do not have any to supply off-hand, but we may see some as we move into incidence geometry.

An easier question to address is: how good is the bound appearing in Theorem 0.3? Do we expect that $|E|^{3/2}/|V|^2$ is the correct function of $|E|$ and $|V|$ by which to estimate $cr$? It worked out well in the case of $K_{t,t}$. Let’s argue that this is the correct function to study. If $G$ is any graph on $n$ vertices, then $cr(G) \leq cr(K_n) = \frac{1}{2} \binom{n}{3}$ - check! If $G$ has edge density $\geq \epsilon > 0$ (meaning that $|E(G)|/|E(K_n)| \geq \epsilon$), then this means that $\frac{1}{6\epsilon} |E|^{3/2}/|V|^2 \leq cr(G) \leq \frac{1}{2\epsilon^2} |E|^{3/2}/|V|^2$. Thus, if we restrict our attention to graphs with edge density $\geq \epsilon$ for a fixed value $\epsilon > 0$, then $cr(G) \sim |E|^{3/2}/|V|^2$, where the implied constant in the lower bound depends on $\epsilon$. Hence the bound looks right within the realm of $\epsilon$-dense graphs. Next observe that the inequality possesses a “symmetry”: if substitute $\lambda$ disjoint copies of $G$ into it, then both the left and the right sides scale linearly in $\lambda$. Good bounds tend to exhibit a symmetry like this: again, I refer you to Tao’s wonderful blog post; I particularly enjoyed his treatment of the Cauchy-Schwarz inequality, which we may revisit a few classes hence. So if we restrict our attention to taking any number of copies of an $\epsilon$-dense graph, then we still get an estimate of.
the form \( cr(G) \sim |E|^3/|V|^2 \). Tao remarks: “It is not hard to see that these examples basically cover all possibilities of \(|V|\) and \(|E|\) for which \(|E| \geq 4|V|\). Thus the crossing number inequality cannot be improved except for the constants.” It is interesting to mull over what exactly this means. I’m content at the moment to see that \(|E|^3/|V|^2\) captures the right order of growth of \( cr \) over many families of graphs.