Problem Set n° 2    MAT 885: Graduate Combinatorics    Due Feb 6, 2018

Again, please submit three solutions of which you are the most proud.

The Zarankiewicz problem.

(1) Let \( ex(n) \) denote the maximum number of edges in a bipartite graph with \( n \) vertices in each part and no copy of \( C_4 \). Suppose that \( q \) is a prime power and \( n = q^2 + q + 1 \). Prove that \( ex(n) = n(\sqrt{4n-3}+1)/2 \) by analyzing Reiman’s construction for the finite projective plane \( P^1(\mathbb{F}_q) \) and the proof of the Kővari-Sós-Turán theorem. Using a famous theorem from number theory, show that without any condition on \( n \), \( ex(n) \) is always within a factor of 8 of \( n(\sqrt{4n-3}+1)/2 \).

(2) The Kővari-Sós-Turán theorem asserts that \( ex(n; K_{s,t}) \lesssim s^{1/t}n^{2-1/t} + tn \). Imagine that \( s \) is fixed, like \( s = 5 \). Compare the bounding terms \( n^{2-1/t} \) and \( tn \). Which one dominates as we vary \( n \) as a function of \( t \)? Show that the term \( tn \) is vital, in the sense that for all \( n \), there exists a \( K_{s,t} \)-free graph on \( n \) vertices with \( \sim tn \) edges.

Counting triangles.

(3) For a graph \( G \), let \( \overline{G} \) denote its complement: \( V(\overline{G}) = V, \ E(\overline{G}) = \binom{V}{2} - E(G) \). Also, let \( t(G) \) denote the number of triangles in \( G \).

(a) Prove that
\[
\sum_{x \in V} \deg_G(x) - \frac{1}{2} \sum_{x \in V} \deg_G(x) = \frac{1}{2} \sum_{x \in V} \deg_{\overline{G}}(x) - \frac{1}{2} \sum_{x \in V} \deg_{\overline{G}}(x) - \frac{1}{2} \binom{n}{3}.
\]

As a hint, let \( p(G) \) denote the number of induced two-edge paths in \( G \), and try to express each of the three terms on the right in terms of \( t(G), t(\overline{G}), p(G), \) and \( p(\overline{G}) \).

(b) Prove that in any 2-coloring of the edges of \( K_n \), the number of monochromatic triangles is at least
\[
\frac{n(n-1)(n-5)}{24}.
\]

What does this tell you about parties of 6 people?

Ramsey theory.

(4) Determine, with proof, which one of the following two statements is true:

(a) (infinite Ramsey) Build a graph \( K_\omega = (V,E) \), where \( V \) is the set of positive integers and \( E \) consists of all pairs of distinct positive integers. Suppose that the edges of \( K_\omega \) are colored red and blue. Then there exists a monochromatic complete subgraph of \( K_\omega \) on infinitely many vertices, i.e. a monochromatic copy of \( K_\omega \).

(b) (infinite van der Waerden) In every 2-coloring of the positive integers, there exists a monochromatic infinite arithmetic progression.

(5) Prove that in any 3-coloring of the numbers 1, \ldots, 10^{17000}, there exists a 3-term mono-\( \chi \) AP.

(6) Show that in any \( c \)-coloring of the edges of \( K_r, r \geq \lceil e \cdot c! \rceil + 1 \), there exists a mono-\( \Delta \) subgraph.

(7) Let \( r_3(6,n) \) denote the smallest value \( r \) such that in any red- / blue- coloring of the hyperedges (tredges?) of \( K_3^n \), there exists either a monochromatic red \( K_3^3 \) or a monochromatic blue \( K_3^n \). Thus, the target substructures have different sizes, based on which color we are considering. First, convince yourself that \( r_3(6,n) \) exists, based on Ramsey’s theorem. Then prove that \( r(n) \leq r_3(6,n) \).

(Problem credit: Kiran Kedlaya.)

(8) Deduce Gallai’s theorem from the Hales-Jewett theorem. You should mimic the deduction of van der Waerden’s theorem. As a hint, it is first useful to translate the given set \( S \) so that all of its coordinates are positive. Then select a value \( L \) larger than the largest coordinate of any element of \( S \). If \( S = \{s_1, \ldots, s_l\} \), map \([i]\)\(^d\) to \( \mathbb{R}^k \) by sending a point of the box \((i_1, \ldots, i_d)\) to the point \( \sum_{j=1}^d s_j \cdot L^j \) in \( \mathbb{R}^k \).