Problem Set nº 4 MAT 885: Graduate Combinatorics Due Feb 22, 2018

Please submit three solutions.

**Mycielski’s construction.**
Suppose that $\chi(G) = k$ and $|G| = n$. Take a copy of $G$, and for every vertex $v$ in $G$, clone it to get a vertex $w$ that has the same neighbors in $G$ as $v$, for a total of $n$ clone vertices. Join a new vertex by edges to all of the clones. Prove that the resulting graph $H$ is triangle-free if $G$ is, and $\chi(H) = \chi(G) + 1$.

**Tutte’s construction.**
Suppose that $\chi(G) = k$ and $|G| = n$. Take $(n-1)^k + 1$ copies of $G$ and a set of $(n-1)k + 1$ new vertices. For every $n$-subset of the new vertices, join them by an arbitrary perfect matching to a distinct copy of $G$. Prove that the resulting graph $H$ has girth 6 if $G$ does, and $\chi(H) = \chi(G) + 1$.

**Erdős’s proof.**
Review Erdős’s proof that there exist graphs of arbitrarily large girth and chromatic number. Think about whether the alteration method is actually necessary. What happens if we choose $p$ so that the probability that $G$ contains no cycles of length $\leq l$ is $> 1/2$? That is, how small must we make $p$? How does this influence the estimate we get on $\alpha(G)$? That is, how small can you make $x$ so that $\Pr(\alpha(G) < x) > 1/2$ for $n \gg 0$? Can you finish the proof, or do you get stuck? What if instead you choose $p$ so that the expected number of cycles of length $\leq l$ is $< 1/2$ and then apply Markov’s theorem?

**Brickyards.**
Find a drawing of $K_{s,t}$ in the plane with $\left\lfloor \frac{s}{2} \right\rfloor \cdot \left\lfloor \frac{s-1}{2} \right\rfloor \cdot \left\lfloor \frac{t}{2} \right\rfloor \cdot \left\lfloor \frac{t-1}{2} \right\rfloor$ crossings.

**Improved crossing numbers.**
Prove that if $G = (V,E)$ is a graph and $|E| - 3|V| + 6 > 0$, then any drawing of $G$ contains a crossing between edges with disjoint endpoints. Use this to prove that the number of crossings in $G$ between edges with disjoint endpoints is at least $|E| - 3|V|$.

**Crossing numbers.**
Suppose that you wish to show that any graph $G = (V,E)$ with $|E| \geq 3.01|V|$ has a large crossing number. Prove a bound of the form $cr(G) \geq c|E|^3/|V|^2$ for some fixed constant $c > 0$. What value do you find for $c$?

**Crossing numbers and multigraphs.**
This sequence of exercises comes from Guth’s book. It involves a pleasant use of the probabilistic method. A multigraph is a graph in which parallel edges may appear. The number of edges parallel to a given edge (including itself) is called the multiplicity of that edge. Suppose that $G = (V,E)$ is a multigraph with maximum edge multiplicity $M$, and suppose that $|E| \geq 4M|V|$.

(a) Prove that

$$cr(G) \geq \frac{1}{64} \frac{|E|^3}{|V|^2 M^3}.$$  

Note that there is no restriction on how parallel edges are drawn: two parallel edges may cross completely different subsets of the other edges of $G$.

(b) Next, assume that each edge multiplicity lies between $M/2$ and $M$. Prove in this case that

$$cr(G) \geq \frac{1}{256} \frac{|E|^3}{|V|^2 M}.$$  

Thus, the constant is a bit worse, but we have made a significant gain in case $M$ is large. (Hint: select one edge at random from each parallelism class in $G$ to form a simple graph $G'$. Express the expected number of crossings in $G'$ in terms of $G$, and apply the crossing number lemma to $G'$.)
(c) Finally, with no condition on minimum edge multiplicities, prove that as long as $|E| \geq 100M|V|$, 

$$cr(G) \geq c\frac{|E|^3}{|V|^2M},$$

for some absolute constant $c > 0$. (Hint: Condition on whether or not more than $1/10$ of the edges have multiplicity $\geq M/2$. If so, apply part (b), and if not, induct on $|E|$.) What value for $c$ do you find works?