Please submit three solutions.

We use the following notation:

- $S$: a compact, connected, orientable surface
- $S_g^n$: the surface $#^g T^2 #^n D^2$; $S_g = S_0^0$
- $A$: a collection of essential, simple arcs on $S$ that are pairwise non-isotopic
- $\Gamma$: a collection of essential, simple closed curves on $S$ that are pairwise non-isotopic

(1) Suppose that $S = S_g^n$ and the curves in $\Gamma$ are pairwise disjoint. What is the maximum cardinality of $|\Gamma|$?

(2) Find a 1-system of arcs on $S_0^n$ of cardinality $(\binom{n-1}{2})$ with the additional feature that one of the boundary components of $S_0^n$ contains all of the endpoints of the arcs.

(3) Suppose that no arc in $A$ intersects more than $r$ other arcs of $A$. Prove that $|A| \leq 3(r + 1)|\chi(S)|$. Examine the special case that $r = 1$. Can you improve the bound on $|A|$? Try to determine the maximum cardinality of $A$ for some simple surfaces, like $S_0^4$ or $S_1^4$.

(4) Suppose that $S = S_g$ and the curves in $\Gamma$ are pairwise disjoint and homologous (mod 2). Correct the following statement: “a pair of disjoint simple closed curves are homologous (mod 2) iff their complement is disconnected.” Using the correct statement, what is the maximum cardinality of $|\Gamma|$?

(5) Suppose that $\Gamma$ is a 1-system of curves on $S_g$. Prove that $|\Gamma| \leq (g - 1)2^{2g}$.

(6) Let $f(g, n)$ denote the maximum cardinality of a 1-system on $S_g^n$. Prove that $f(g, n) \leq f(g, n - 1) + (2g + 1)$. Hint: given a 1-system on $S_g^n$, cap off one of the boundary components, and study the resulting curves on $S_{g-1}^{n-1}$. How could they fail to be a 1-system?

(7) In preparation for (Linear) Independence Day, the $n$ inhabitants of Oddtown form clubs, as usual, but they adhere to the unusual ordinance that any club must contain an even number of members, while any two distinct clubs must contain an odd number of members in common. What is the maximum number of clubs they can form?