Please submit three solutions.

(1) Recall or look up the statement of Hall’s matching theorem. Use it to prove that if \( G = (V, E) \) is a bipartite graph with bipartition \( V = A \sqcup B \), \(|A| \leq |B|\), every vertex in \( A \) has the same degree, and every vertex in \( B \) has the same degree, then there exists a matching from \( A \) to \( B \). Carefully explain how to prove Sperner’s theorem on antichains using this result.

(2) In this exercise, we prove:

**Theorem** (Erdős-Ko-Rado): If \( \mathcal{H} \) is a hypergraph on \([n] = \{1, \ldots, n\}\) in which each edge has cardinality \( k \) and each pair of edges intersects non-trivially, then \(|\mathcal{E}(\mathcal{H})| \leq \binom{n-1}{k-1}\).

The proof is an application of the permutation method.

- Call a cyclic permutation \( \pi \) of \([n]\) *compatible* with an edge \( e \in \mathcal{E}(\mathcal{H}) \) if the elements of \( e \) occur consecutively with respect to \( \pi \). For instance, if \( e_1 = \{1, 3, 4, 6\} \) and \( e_2 = \{2, 6, 7\} \) are edges of \( \mathcal{H} \) and \( \pi \) is the cyclic permutation \( (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7)(6 \ 2 \ 5 \ 7 \ 1 \ 4 \ 3) \), then \( \pi \) is compatible with \( e_1 \) but not \( e_2 \).
- Given a cyclic permutation \( \pi \), show that \( \pi \) is compatible with at most \( k \) edges of \( \mathcal{H} \).
- Given an edge \( e \in \mathcal{E}(\mathcal{H}) \), how many cyclic permutations \( \pi \) are compatible with \( e \)?
- Complete the proof of the theorem using what you have just proven.

(3) Suppose that \( \gamma_1, \ldots, \gamma_k \) are curves on \( S_g \) pairwise intersecting exactly once. The Malestein-Rivin-Theran theorem states that \( k \leq 2g + 1 \). Prove that \( k = 2g + 1 \) iff the (mod 2) homology classes of the curves are linearly dependent.

(4) Establish the triangular criterion: suppose that \( F \) is a field, \( v_1, \ldots, v_k \in F^n \), and \( f_1, \ldots, f_k \in F[x_1, \ldots, x_n] \) are polynomials such that \( f_i(v_j) = 0 \) if \( j < i \) and \( f_i(v_i) \neq 0 \). Prove that \( f_1, \ldots, f_k \) are linearly independent.

(5) In this exercise, we prove:

**Theorem** (Ray-Chaudhuri-Wilson): If \( L \) is a set of \( n \) non-negative integers, and \( \mathcal{H} \) is an \( L \)-intersecting hypergraph on \([n] = \{1, \ldots, n\}\) with \( k \) edges, then \( k \leq \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} \).

- Begin by writing \( \mathcal{E}(\mathcal{H}) = \{e_1, \ldots, e_k\} \), where the edges are arranged in increasing order of size: \(|e_1| \leq |e_2| \leq \cdots \leq |e_k|\).
- Let \( v_1, \ldots, v_k \in \{0, 1\}^n \) denote the indicator vectors of \( e_1, \ldots, e_k \), respectively.
- Define polynomials \( f_1, \ldots, f_k \in F[x_1, \ldots, x_n] \) by setting \( f_j(x) = \prod_{l \in L, l < |e_j|} l(x - v_j) \).
- Use the triangular criterion to prove that \( f_1, \ldots, f_k \) are linearly independent.
- Mimic the proof of the modular version of the Ray-Chaudhuri-Wilson theorem to complete the argument.

(6) A \( k \)-distance set \( A \subset \mathbb{R}^n \) is a set with the property that there exist \( k \) values \( d_1, \ldots, d_k \) such that the distance \( d(a, b) \) between any pair of distinct elements \( a, b \in A \) belongs to the set \( \{d_1, \ldots, d_k\} \).

- Determine, with proof, the maximum cardinality of a 1-distance set \( A \subset \mathbb{R}^n \).
- Let \( A \subset \mathbb{R}^n \) denote the set of all indicator vectors of 2-element subsets of \([n]\). Prove that \( A \) is a 2-distance set. What is \(|A|\)?
- Following Blokhuis, prove that if \( A = \{a_1, \ldots, a_k\} \subset \mathbb{R}^n \) is a 2-distance set, then \(|A| \leq (n+1)(n+4)/2\), as follows. Suppose that \( d(a_i, a_j) \in \{d_1, d_2\} \) for all pairs of distinct \( a_i, a_j \in A \). Define polynomials

\[
f_j(x) = (d(x, a_j)^2 - d_1^2)(d(x, a_j)^2 - d_2^2), \quad j = 1, \ldots, k.
\]

Prove that \( f_1, \ldots, f_k \) are linearly independent.
- Prove that \( f_j(x) \) belongs to the span of the polynomials

\[
1; \quad x_i, 1 \leq i \leq n; \quad x_i x_j, 1 \leq i < j \leq n; \quad x_i \cdot \left( \sum_{j=1}^n x_j^2 \right), 1 \leq i \leq n; \quad \left( \sum_{j=1}^n x_j^2 \right)^2.
\]

• Use the preceding facts to derive the desired bound on $|A|$.
• For extra credit, show that the polynomials $1, x_1, \ldots, x_n, f_1, \ldots, f_k$ are linearly independent, and thereby obtain a stronger bound on $|A|$.

(7) Suppose that $A$ is a symmetric $n \times n$ matrix such that
• all diagonal entries are positive,
• all off-diagonal entries are non-positive,
• the sum of the entries in any row or column is non-positive, and
• the sum of the entries in some row or column is positive.
Prove that $\text{rk}(A) \geq n$. In fact, show that $x^T A x \geq 0$ for any $x \in \mathbb{R}^n$, with equality if and only if $x = 0$.

(8) Suppose that $\lambda \geq 2$. For infinitely many values $n$, describe an example of a $\lambda$-intersecting hypergraph on $[n]$ with $n$ edges.