Please submit three solutions.

Mod prime power town.

(1) Suppose that $q$ is a prime power and that $C_1, \ldots, C_m$ are subsets of $\{1, \ldots, n\}$ with the property that $|C_i \cap C_j| \equiv 0 \pmod{q}$ if and only if $i \neq j$. Prove that $m \leq n$. As a hint, apply the usual technique, but work over $\mathbb{Q}$. The coefficients in a linear dependence are rational numbers, which you can rescale to be relatively prime integers.

The permutation method.

(2) Prove the following theorem of Bollobás. Let $A_1, B_1, \ldots, A_m, B_m$ be subsets of $\{1, \ldots, n\}$ with the property that $|A_i| = p$ for all $i = 1, \ldots, m$, $|B_i| = q$ for all $i = 1, \ldots, m$, and $A_i \cap B_j = \emptyset$ iff $i = j$. Prove that $m \leq \binom{p+q}{p}$, and that the bound is tight for all $p$ and $q$. (Note that the bound is independent of $n$.) As a hint, think about how to define a notion of compatibility between a permutation of $\{1, \ldots, n\}$ and a set $A_i$, similar to what we did in the proofs of Sperner’s theorem and the Erdős-Ko-Rado theorem.

(Note: there is a very interesting algebraic proof of this result, and a generalization thereof, due to Lovász.)

Oldies but goodies.

(3) Look through our past problem sets for any problems you wish had worked on, but didn’t, or did not feel satisfied with your earlier solution to.