Floer homology and Dehn surgery

Josh Greene

Boston College

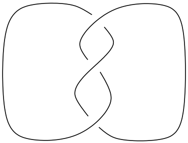
Thirty years of Floer homology for 3-manifolds
Casa Matemática Oaxaca

August 1, 2017

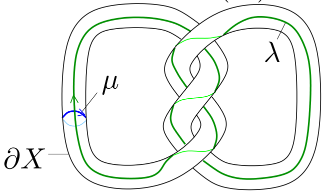
Canonical front matter.

Dehn surgery was introduced by Max Dehn (1910) as a method for generating 3-manifolds from knots and links.

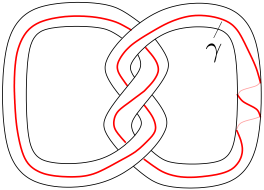
$K = T_{2,3} \subset S^3$:



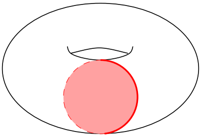
$X = X_K := S^3 \setminus \nu(K)$:



$X(\gamma)$, $\gamma \subset \partial X$ a **slope**:



gl$\bigcup$e



$S^1 \times D^2$

If $[\gamma] = p[\mu] + q[\lambda] \in H_1(\partial X; \mathbb{Z})/\pm$, then write $K(p/q) = X(\gamma)$.
The preceding example depicts $T_{2,3}(+1)$, the famous Poincaré homology sphere.
The construction is ubiquitous:

Theorem 1 (Lickorish, Wallace 1960)

Every closed, oriented 3-manifold is the result of Dehn surgery along a link in S^3.

Of particular interest is the case of *knots*.

Questions.

Here are some natural questions one may consider:

- **Realizability:** which 3-manifolds arise by Dehn surgery along a knot in S^3?
- **Classification:** what are all ways to obtain a fixed 3-manifold by Dehn surgery along a knot in S^3?
- **Exceptional surgeries:** which surgeries do not carry a hyperbolic structure (e.g. how do you get lens spaces and reducible manifolds)?
- **Cosmetic surgery:** when can different surgeries along the same knot produce the same result?
- **Characterizing slopes:** when does the pair $(K(p/q), p/q)$ determine K?

Many other problems in low-dimensional topology abut to these ones (e.g. unknotting number and concordance).

Techniques.

There exist a variety of techniques for studying problems in Dehn surgery:

- combinatorial methods;
- representation / character varieties of 3-manifold groups;
- sutured manifolds and taut foliations;
- hyperbolic geometry;
- gauge theory; and
- **Floer homology**: attacks on classical problems and new lines of inquiry; often provides complementary information to the other techniques.

Floer homology.

We will focus on the last half of thirty years of Floer homology for 3-manifolds; and we will also focus almost exclusively on Heegaard Floer homology.

Let's quickly swipe through the following slides.

Background on Floer homology.

3-manifolds.

"Classical" Heegaard Floer homology was defined by Ozsváth-Szabó (c. 2000). It assigns invariants $HF^\circ(Y,s)$ to a closed, oriented pointed 3-manifold Y equipped with a spin^c structure s, where $\circ \in \{+,-,\hat{},\infty\}$. The invariant $\widehat{HF}(Y,s)$ is a finitely generated $\mathbb{Z}$-module, while the other three are finitely generated $\mathbb{Z}[U]$-modules. The invariants are related by various long exact sequences:

$$\cdots \to HF^+(Y,s) \xrightarrow{\cdot U} HF^+(Y,s) \to \widehat{HF}(Y,s) \to \cdots$$

$$\cdots \to HF^-(Y,s) \to HF^\infty(Y,s) \xrightarrow{\pi} HF^+(Y,s) \to \cdots$$

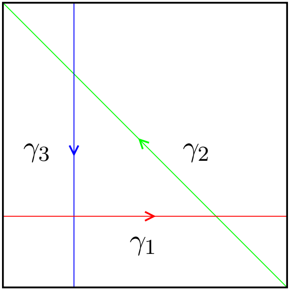
For a torsion spin^c structure s, $HF^\infty(Y,s) \approx \mathbb{Z}[U,U^{-1}]$. Moreover, $\mathrm{im}(\pi) \approx \mathcal{T}^+ := \mathbb{Z}[U,U^{-1}]/U\cdot\mathbb{Z}[U]$, and $\mathrm{coker}(\pi) =: HF^{red}(Y,s)$ is a finitely generated $\mathbb{Z}$-module.

Assuming s is torsion (e.g. if Y is a rational homology sphere), each invariant carries a relative $\mathbb{Z}$-grading that the maps in the sequences change in a controlled way; e.g., the U-action lowers it by 2. Furthermore, the relative $\mathbb{Z}$-grading can be enhanced to an *absolute* $\mathbb{Q}$-grading. The loss of intuition for what it means ("what is a $-17/4$-dimensional class?") is balanced by a wealth of information, which we shall see.

Of particular emphasis is the d-**invariant** $d(Y, s) \in \mathbb{Q}$: this is the minimum grading of a non-zero element in $\mathrm{im}(\pi) \subset HF^+(Y, s)$.

The surgery exact triangle.

Suppose that X is a knot exterior and $(\gamma_1, \gamma_2, \gamma_3)$ is a triple of oriented slopes on ∂X satisfying $\gamma_1 \cdot \gamma_2 = \gamma_2 \cdot \gamma_3 = \gamma_3 \cdot \gamma_1 = 1$.



The triple of manifolds $(X(\gamma_1), X(\gamma_2), X(\gamma_3))$ forms a **triad**. Their invariants obey an exact triangle

$$\cdots \to HF^\circ(X(\gamma_1)) \to HF^\circ(X(\gamma_2)) \to HF^\circ(X(\gamma_3)) \to \cdots$$

Exact triangles like this were first discovered by Floer in the context of instanton Floer homology.

An important special case arises when X is the exterior of a knot $K \subset S^3$, $\gamma_1 = 1/0$, and γ_2 and γ_3 are consecutive integers:

$$\cdots \to HF^\circ(S^3) \to HF^\circ(K(p)) \to HF^\circ(K(p+1)) \to \cdots$$

4-manifolds and TQFT structure.

If W is a cobordism from Y_1 to Y_2, and s is a spin^c structure on W restricting to s_i on Y_i, then there exists a map

$$F^{\circ}_{W,s} : HF^{\circ}(Y_1, s_1) \to HF^{\circ}(Y_2, s_2).$$

Grading shift: $F^{\circ}_{W,s}$ shifts grading by the amount

$$\frac{c_1(s)^2 - 2\chi(W) - 3\sigma(W)}{4}$$

(reminiscent of a dimension of a moduli space in SW). If W is negative definite and $Y_1 = S^3$, this leads to bounds on the d-invariant of ∂Y_2 in terms of $(H_2(W;\mathbb{Z}), Q_W)$.

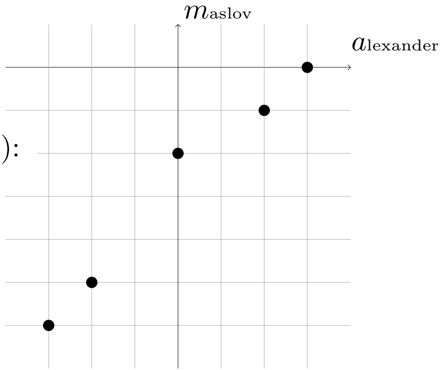
Knot Floer homology.

Knot Floer homology was defined by Ozsváth-Szabó and J. Rasmussen (c. 2003). It assigns invariants to pointed knots and links. The simplest version is the invariant $\widehat{HFK}(K)$ for $K \subset S^3$, which is a finitely generated bigraded $\mathbb{Z}$-module.

$\widehat{HFK}(T_{3,4}) =$

$\bigoplus_{m,a} \widehat{HFK}_m(T_{3,4}, a)$:

$(\bullet = \mathbb{Z})$



Everyone's favorite facts about knot Floer homology.

It categorifies the Alexander polynomial:

Theorem 2 (Ozsváth-Szabó, Rasmussen 2004)

$\Delta_K(T) = \sum_{a,m} (-1)^m \cdot \mathrm{rk}\, \widehat{HFK}_m(K,a) \cdot T^a.$

It detects the Seifert genus:

Theorem 3 (Ozsváth-Szabó 2004)

$g(K) = \max\{a \mid \bigoplus_m \widehat{HFK}_m(K,a) \neq 0\}.$

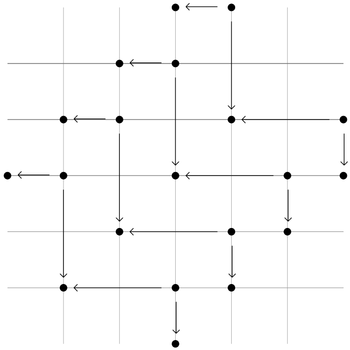
And it detects whether a knot fibers:

Theorem 4 (Ghiggini for $g=1$ 2008, Ni 2007, Juhász 2008)

$\bigoplus_m \widehat{HFK}_m(K, g(K)) \approx \mathbb{Z} \iff K$ *fibers.*

A more general version is the invariant $CFK^\infty(K)$, the chain homotopy type of a $(\mathbb{Z} \oplus \mathbb{Z})$-filtered chain complex over $\mathbb{Z}[U]$.

$CFK^\infty(T_{3,4})$:



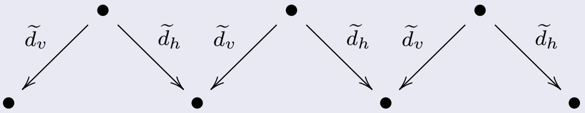
$CFK^\infty(K)$ determines $HF^\circ(K(p/q))$ by a more or less algebraic procedure, the **mapping cone formula**. For example, if $g = g(K) > 1$, then $HF^+(K(0), g-1) \approx \widehat{HFK}(K, g)$.

L-spaces and L-space knots.

- An **L-space** is a rational homology sphere with the simplest Floer homology: $\operatorname{rk} \widehat{HF}(Y, s) = 1$ for all spin^c structures s.
- An **L-space knot** is a knot in S^3 with a non-trivial surgery to an L-space.

Theorem 5 (Ozsváth-Szabó 2006)

A knot K is an L-space knot iff $(\widehat{HFK}(K), \widetilde{d}_v, \widetilde{d}_h)$ is a **chain**:



Essential features of HF for applications:

- $HF^\circ(Y, s)$ carries a lot of algebraic structure. The correction term a.k.a. d-invariant $d(Y, s)$ alone carries very powerful information.
- A filling of Y by a definite 4-manifold X gives bounds on $d(Y, s)$ in terms of the intersection pairing on $(H_2(X; \mathbb{Z}), Q_X)$.
- Floer homology groups of a triad obey an exact triangle.
- Knot Floer homology captures basic topological features of a knot: its Alexander polynomial, genus, and fiberedness.
- The knot Floer homology of K governs the Floer homology of Dehn surgeries along K by mapping cone formulae.
- The situation is especially nice if K is an L-space knot.

Onto the problems!

Prehistory: first cases of Dehn surgery characterizations.

Theorem 6 (Dehn surgery characterization of S^3; Gordon-Luecke 1989)

If $K(p/q) \approx S^3$, then either $p/q = 1/0$ or $K \simeq U$.

The method of proof is primarily combinatorial in nature. Theorem 6 has an immediate corollary:

Theorem 7 (The knot complement problem)

Knots in S^3 with orientation-preserving homeomorphic exteriors are isotopic.

Theorem 8 (Property R; Gabai 1987)

If $K(p/q) \approx S^1 \times S^2$*, then* $K \simeq U$ *and* $p/q = 0/1$*. Moreover, if* $K(0)$ *is reducible, then* $K \simeq U$*.*

The method of proof involves sutured manifolds and taut foliations.
Gordon-Luecke reproved it by combinatorial methods.

"Easy" HF proof of the first statement!

If $g = g(K) > 1$, then $HF^+(K(0), g-1) \approx \widehat{HFK}(K, g) \neq 0$. On the other hand, $HF^+(S^1 \times S^2, i) = 0$ for $i \neq 0$. For $g = 1$, run a similar argument with twisted coefficients. Conclude that $g = 0$, so $K \simeq U$. □

The rub: deep results about sutured manifolds, etc., go into establishing the basic features of HF used here, e.g. genus detection.

Gordon conjectured the following result, which subsumes Theorems 6 and the first part of 8:

Theorem 9 (Kronheimer-Mrowka-Ozsváth-Szabó 2005)

If $K \subset S^3$, $r \in \mathbb{Q}$, and $K(r) \approx U(r)$, then $K \simeq U$.

The proof of Theorem 9 relies on the case $r = 0$, i.e. Property R, and the exact triangle. A repackaging of the proof: K is an L-space knot; correction terms imply that its Alexander polynomial is 1; hence $g(K) = 0$ and $K \simeq U$.

Realizability.

Given a target closed, oriented 3-manifold Y, is Y the result of Dehn surgery along a knot in S^3?

Homological obstruction: If $K \subset S^3$, then $H_1(K(p/q)) \approx \mathbb{Z}/p\mathbb{Z}$. E.g. $\mathbb{R}P^3 \# \mathbb{R}P^3$ is not realizable.
π_1 *obstruction:* $\pi_1(K(p/q))$ has *weight one*: it can be killed by introducing a single relation, viz. $[\mu_K] = 1$.

Theorem 10 (Gordon-Luecke 1987)

If Y is a reducible Dehn surgery along a knot in S^3, then Y contains a lens space summand.

The proof is based (largely) on combinatorial methods. Consequently, a reducible integer homology sphere is not realizable.

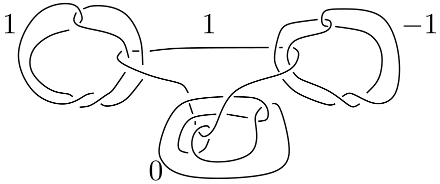
Problem 1 (Kirby Problem 3.6(c))

Does there exists an irreducible homology sphere that is not realizable?

Theorem 11 (Auckly 1993)

There exists a hyperbolic homology sphere that is not realizable.

Auckly first answered Kirby's problem with an example of a graph manifold:



The proof that it is non-realizable uses gauge theory, viz. Taubes's variation on Donaldson's theorem for a manifold with periodic ends. (Note the presence of a definite 4-manifold.) The enhancement to a hyperbolic example uses cobordisms written down by Livingston and Myers.

These were the only two examples in the literature until recently!
(However, Auckly certainly knew how to construct infinitely many examples.)

Theorem 12 (Hom-Karakurt-Lidman 2015)

There exist infinitely many small Seifert fibered spaces that are not realizable. More precisely, for $p \geq 4$, the Brieskorn spheres $Y_p = \Sigma(2p, 4p-1, 4p+1)$ are not realizable. Moreover, their fundamental groups have weight one.

They proved it by way of the following fact:

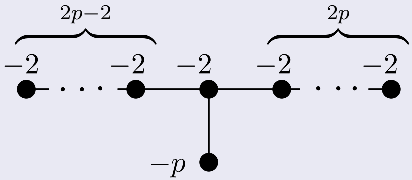
Theorem 13 (HKL 2015)

If $d(K(1/n)) \leq -8$, then $U \cdot HF_0^{red}(K(1/n)) \neq 0$.

The proof of Theorem 13 relies on the mapping cone formula relating $CFK^\infty(K)$ to $HF^+(K(1/n))$.

Sketch of Proof of Theorem 12.

Y_p is the boundary of X_p, a negative definite plumbing of disk bundles over spheres:



Ozsváth-Szabó used the exact triangle and maps induced by cobordism to show that $(H_2(X;\mathbb{Z}), Q_X)$ determines $HF^+(Y)$. The invariant extracted from $(H_2(X;\mathbb{Z}), Q_X)$ is called *lattice homology*. Its computation was subsequently developed by Némethi, Karakurt, et al. One shows that $d(Y_p) = -4p$ and $U \cdot HF_0^{red}(Y_p) = 0$. By Theorem 13, Y_p is not realizable. □

And, in fact:

Theorem 14 (Hom-Lidman 2016)

There exist infinitely many hyperbolic homology spheres that are not realizable.

The examples are surgeries along a hyperbolic genus-one knot in the connected sum of several copies of the Poincaré homology sphere.
After the talk, I got the following excellent question:

Question 1 (Michel Boileau)

For every natural number n, does there exist an integer homology sphere that is not surgery along a link in S^3 of n or fewer components? What about $n = 2$?

Perhaps the link surgery formulae in HF can provide an approach.

Exceptional surgeries.

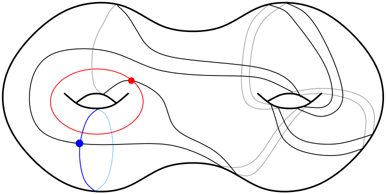
Theorem 15 (Thurston 1976)

All but finitely many Dehn fillings on a hyperbolic knot exterior are hyperbolic.

When does a hyperbolic knot exterior admit more than one non-hyperbolic filling?
As a special instance, which knots in S^3 admit a surgery to a lens space or a reducible manifold?

Lens spaces.

Berge's construction.
Suppose that $\Sigma \subset S^3$ is a genus-2 Heegaard surface that cuts S^3 into a pair of handlebodies U_1, U_2, and there exist meridian disks $D_i \subset U_i$ each meeting a knot $K \subset \Sigma$ in a single transverse point of intersection.



Surface-framed surgery along K is a lens space.

Berge described several families of such *doubly-primitive* knots and a corresponding list

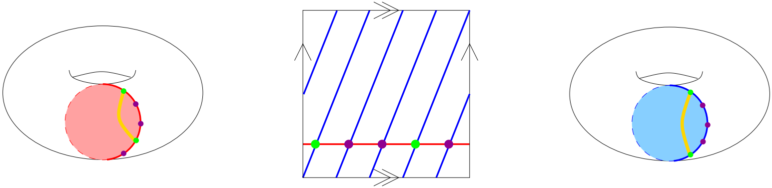
$$\mathcal{B} = \{(p,q) \mid L(p,q)\,\text{arises by this construction}\}.$$

Conjecture 16 (Berge conjecture 1990)

A knot in S^3 with an integer surgery to a lens space arises by Berge's construction. In particular, if $L(p,q)$ is realizable, then $(p,q) \in \mathcal{B}$.

Berge furthermore observed that the surgery dual to a doubly-primitive knot has a particularly pleasant form: it is a **simple knot**.

A pictorial definition / example:



At left and right are two solid tori with meridional disks highlighted.
Their boundaries are identified with the torus in the middle to produce the lens space $L(5,2)$.
There are five points of intersection between the red and blue curves, and the simple knot $K(5,2,3)$ is the union of the yellow arcs in the two disks that run between the two green points.

Conjecture 17 (Berge conjecture reformulated)

If a knot in a lens space admits an integer slope surgery to S^3, then it is a simple knot.

Obstructions to lens space surgery.

Theorem 18 (Cyclic Surgery Theorem; Culler-Gordon-Luecke-Shalen 1987)

If a knot exterior admits two fillings with cyclic fundamental group, then the filling slopes have distance one, or else the knot exterior is Seifert-fibered.

Corollary 19

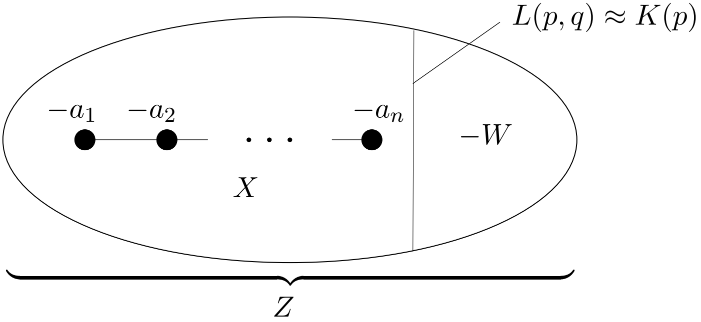
If a knot in S^3 admits a lens space surgery, then either the knot is a torus knot, or else the surgery slope is an integer.

Homological obstruction: If $L(p,q)$ is realizable by an integer slope surgery, then $q \equiv -\Box \pmod{p}$ (Fintushel-Stern).

Since lens spaces are L-spaces, a knot with a lens space surgery must be fibered (Theorems 4 and 5), and the non-zero coefficients of its Alexander polynomial must be ± 1 and alternate in sign (Theorem 5).

Gauge theoretic obstruction.

- $L(p,q)$ bounds a negative definite plumbing of disk bundles along spheres, $X = X(p,q)$.
- Dehn surgery $K(p)$, $p > 0$, bounds a positive definite 4-manifold $W = W(K,p)$ with a single 0- and 2-handle.
- Assume that $L(p,q) \approx K(p)$. Form $Z := X \cup -W$.



Z is a closed, smooth, negative definite 4-manifold; hence:

Theorem 20 (Donaldson's Theorem A)

The intersection pairing on Z is diagonalizable over $\mathbb{Z}$.

Write $\Lambda(p,q) := (H_2(X(p,q);\mathbb{Z}), Q_X)$.

Corollary 21

If $L(p,q)$ is realizable, then $\Lambda(p,q) \hookrightarrow \mathbb{Z}^{n+1}$.

Corollary 21 is not a perfect obstruction; e.g. $\Lambda(10,1) \hookrightarrow \mathbb{Z}^2$, but $L(10,1)$ is not realizable.

HF to the rescue.

The d-invariant leads to an enhancement.

A vector $\sigma = (\sigma_1, \ldots, \sigma_{n+1}) \in \mathbb{Z}^{n+1}$ is a **changemaker** if, given coins of values $|\sigma_1|, \ldots, |\sigma_{n+1}|$, it is possible to make exact change in any amount from 1 up to the total value of the coins (i.e. $\|\sigma\|_1$). The orthogonal complement of σ is a **changemaker lattice**.

Theorem 22 (Lens space realization; G 2013)

1. *If $K(p) \approx L(p,q)$, then the lattice $\Lambda(p,q)$ is a changemaker lattice.*
2. *If $\Lambda(p,q)$ is a changemaker lattice, then $(p,q) \in \mathcal{B}$.*

The proof of the second part of the theorem involves a detailed combinatorial analysis.

S. Rasmussen, in her 2009 thesis, and Tange (2010) independently solved special cases of the realization problem, also by applying the d-invariant. In particular, S. Rasmussen solved the realization problem under a natural assumption on the homology class of the dual knot in $L(p, q)$.

Other spherical manifolds.

Spherical manifolds fall into types: **C**, **T**, **O**, **I**, **D**.
Type **C** is an alias for lens spaces.
The realization problem for types **T**, **O**, **I** was settled by Gu (2014) using the d-invariant (no lattice enhancement required).
The remaining type, **D**, consists of prism manifolds $P(p, q)$ and is the most abundant. A construction of Berge-Kang conjecturally accounts for all Dehn surgery descriptions of these manifolds.

Theorem 23 (Ballinger-Hsu-Mackey-Ni-Ochse-Vafaee 2016)

The prism manifold $P(p, q)$, $p > 1$, $q < 0$, is realizable iff it appears on Berge-Kang's list.

The proof proceeds along the lines of that of Theorem 22.

Between S^3 and the Poincaré homology sphere

The classical case of the Poincaré homology sphere P deserves special mention:

Theorem 24 (Ghiggini 2008)

If $K(p/q) \approx P$, then $p/q = 1$ and $K \simeq T_{2,3}$.

This result follows from Ghiggini's theorem that $\widehat{HFK}$ detects genus-1 fibered knots.

Between the Poincaré homology sphere and lens spaces.

Problem 2

Which lens spaces are realizable by Dehn surgery along a knot in P?

If $K(p) \approx L(p, q)$, then $2g(K) - 1 \leq p$. Inequality is strict for $K \subset S^3$, but it is attained by some examples in P.
Hedden conjectured a classification of the knots attaining equality. Tange listed some families of knots for which equality is strict; they emulate Berge's exceptional types.
The analogue of the realization problem in this setting seems feasible; it involves studying embeddings into $E_8 \oplus \mathbb{Z}^n$.

Between $S^1 \times S^2$ and lens spaces.

Here we have the analogue to the Berge conjecture:

Conjecture 25

If integer surgery along a knot K in a lens space results in $S^1 \times S^2$, then K is a simple knot.

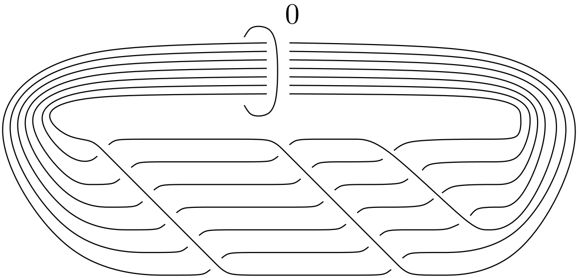
The realization problem was settled by Lisca (2007) and an observation of J. Rasmussen. The homology classes of such knots were conjectured by Baker-Buck-Lecuona (2015).

Theorem 26 (Ni-Vafaee 2017)

A knot in $S^1 \times S^2$ with an L-space surgery is a spherical braid, i.e. it is transverse to each pt. $\times S^2$, *i.e. it fibers with planar surface fibers.*

Cebanu proved Theorem 26 for lens spaces in his 2011 thesis.

Here is a spherical braid:



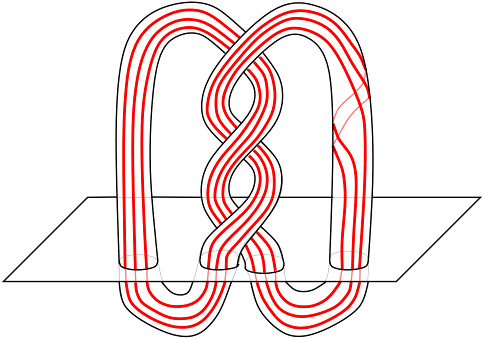
Both 18 and 19 surgery along this knot yield lens spaces, viz. $L(49, 18)$ and $L(49, 19) \approx -L(49, 18)$, respectively. This example, a filling on the famous Berge manifold, is due to Bleiler-Hodgson-Weeks (1999).

A knot K in an L-space Y is **Floer simple** if $\operatorname{rk} \widehat{HFK}(K, Y) = \operatorname{rk} \widehat{HF}(Y) (= |H_1(Y; \mathbb{Z})|)$.
A knot K in a lens space with integer surgery to S^3 or $S^1 \times S^2$ is Floer simple (Hedden, J. Rasmussen 2007).
Thus the following very attractive conjecture subsumes the Berge conjecture and its analogue for $S^1 \times S^2$:

Conjecture 27 (Baker-Grigsby-Hedden; J. Rasmussen 2007)

Floer simple knots in lens spaces are simple.

The cabling construction.



If $K \subset S^3$, then $C_{p,q} \circ K$ is a curve on the torus $\partial\overline{\nu(K)}$. Surgery on $C_{p,q} \circ K$ with respect to the annular slope is $K(p/q)\#L(q,p)$. The construction is very reminiscent of Berge's, and it predated his by a few years.

Conjecture 28 (The cabling conjecture; Gonzalez-Acuña – Short 1986)

If a knot in S^3 admits a reducible surgery, then it arises by the cabling construction.

Theorem 29 (Gordon-Luecke 1986)

If a knot in S^3 admits a reducible surgery, then the surgery slope is an integer, and the result has a lens space summand.

Theorem 30 (G 2015)

If $K(r)$ is a connected sum of lens spaces, then K is a torus knot or a cable thereof.

The proof of Theorem 30 blends complementary genus bounds from Floer homology and combinatorial methods (J. Hoffman and Matignon-Sayari).

Cosmetic surgery.

Let X denote a knot exterior. Two slopes $\gamma_1, \gamma_2 \subset \partial X$ are

- **equivalent** if there exists an orientation-preserving homeomorphism of X taking γ_1 to γ_2;
- **cosmetic** if $X(\gamma_1) \approx \pm X(\gamma_2)$; and
- **truly cosmetic** if $X(\gamma_1) \approx X(\gamma_2)$.

Examples:

1. $U(p/q) \approx U(p/(p+q)) \approx U(p/q')$ for all coprime p, q, and $qq' \equiv 1 \pmod{p}$. The first two slopes are equivalent, but typically not the second two.
2. If K is an amphichiral knot, then $K(r) \approx -K(-r)$ for all r.
3. (Mathieu 1990) $T_{2,3}(9/1) \approx -T_{2,3}(9/2)$, etc.
4. (Bleiler-Hodgson-Weeks 1999) A hyperbolic knot exterior with inequivalent surgeries to $L(49, 18) \approx -L(49, 19)$.

Conjecture 31 (Cosmetic surgery conjecture; Kirby Problem 1.81)

There do not exist truly cosmetic surgeries on inequivalent slopes on an irreducible non-trivial knot exterior.

Theorem 32 (Ozsváth-Szabó 2005)

If $K \subset S^3$ and $K(r) \approx \pm K(s)$, then either $K(r)$ is an L-space or r and s have opposite signs.

Theorem 33 (Wu 2011)

If $K \subset S^3$ is a non-trivial knot and $K(r) \approx K(s)$, then r and s have opposite signs.

Sketch of proof of Theorem 33.

For fixed $p > 0$, the rank of $\widehat{HF}(K(p/q))$ is (more or less) a linear function of $q > 0$ (thanks to the mapping cone formula).
Assume $Y := K(p/q_1) \approx K(p/q_2)$, $p, q_1, q_2 > 0$, $q_1 \neq q_2$.
Then the coefficient on q is 0.
This in turn implies that Y is an L-space.
Thus, K is an L-space knot, and $\Delta''_K(1) \neq 0$, assuming $K \not\approx U$.
On the other hand, the Casson-Walker and Casson-Gordon invariants obey surgery formulae which together imply that if $K(p/q_1) \approx K(p/q_2)$, then $\Delta''_K(1) = 0$, a contradiction. □

Pressing deeper into the mapping cone formulae leads to an improvement:

Theorem 34 (Ni-Wu 2015)

Suppose that K is a non-trivial knot such that $K(r_1) \approx K(r_2)$ with $r_1 \neq r_2$. Then $r_1 = -r_2 = p/q$, where p and q are coprime integers with $q^2 \equiv -1 \pmod{p}$.

Characterizing slopes.

A slope γ is **characterizing** for a knot $K \subset S^3$ if $K(\gamma) \not\approx K'(\gamma)$ for all $K' \subset S^3$, $K' \not\simeq K$.

Examples:

1. Every non-trivial slope is a characterizing slope for the unknot (KMOSz Theorem 9) and for the trefoils and figure-eight knot (Ozsváth-Szabó 2006, buidling on Ghiggini's Theorem 24);
2. (Ni-Zhang) $T_{4,5}(21) \approx T_{2,11}(21) \approx L(21,4)$;
3. (Baker- J. Rasmussen) $4n+3$ is a characterizing slope for $T_{2,2n+1}$;
4. (G Theorem 30) rs is a characterizing slope for $T_{r,s}$.

Characterizing slopes for torus knots.

Theorem 35 (Ni-Zhang 2014)

If $p/q = O((rs)^2)$, then p/q is a characterizing slope for $T_{r,s}$.

Sketch of Proof of Theorem 35.

Suppose that $K(p/q) \approx T_{r,s}(p/q)$, and that $p/q \geq rs$.
Then $T_{r,s}(p/q)$ is an L-space, so K is an L-space knot.
This gives $\Delta''_K(1) \geq 2(2g - 1)$.
The surgery formula for the Casson-Walker invariant (Boyer-Lines) gives $\Delta''_K(1) = \Delta''_{T_{r,s}}(1) = O((rs)^2)$.
Hence $g(K) = O((rs)^2)$.
If K is hyperbolic, then the 6-theorem leads to a bound $p = O(g)$; thus $p/q = O(g) = O((rs)^2)$, as claimed.
If K is not hyperbolic, then more work is involved (e.g. an appeal to Gabai's work on knots in solid tori). □

Related techniques lead to an improvement:

Theorem 36 (McCoy 2016)

If $p/q = O(rs)$, then p/q is a characterizing slope for $T_{r,s}$. Moreover, $T_{r,s}$ has only finitely many non-characterizing slopes apart from negative integers.

Conjecturally, the condition on negative integers can be removed.

Theorem 37 (Lackenby 2017)

Every knot has a characterizing slope.

L-space knots.

Problem 3

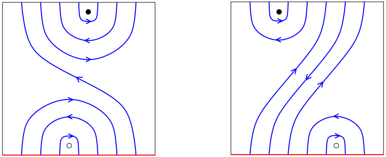
Characterize L-space knots topologically.

There is no guiding conjecture, apart from a reduction to the L-space conjecture.

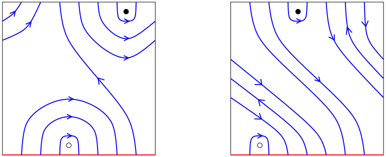
L-space knots...

- fiber (Ozsváth-Szabó, Ni);
- carry the standard tight contact structure (Hedden, Ni);
- are strongly quasi-positive (Hedden);
- are braided, if they are satellites (Baker-Motegi $\geq$ 2017)

A knot in S^3 or a lens space is a $(1,1)$ **knot** if it is 1-bridge with respect to the genus-1 Heegaard splitting.



two incoherent diagrams of 5_2, and
two coherent diagrams of $T(2,7)$



Theorem 38 (G-Lewallen-Vafaee 2016)

A $(1,1)$ diagram D presents an L-space knot iff D is coherent.

Corollary 39

1-bridge braids in lens spaces are L-space knots.

El fin.