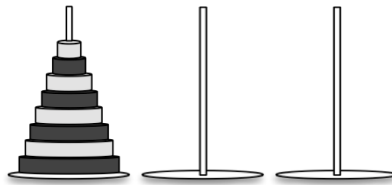


ON THE FRAME-STEWART ALGORITHM FOR THE TOWER OF HANOI

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1. INTRODUCTION

The Tower of Hanoi puzzle was created over a century ago by the number theorist Edouard Lucas [2, 4], and it and its variants have challenged children as well as professional mathematicians and computer scientists ever since. The puzzle consists of three vertical pegs attached to a base, and a number of disks of different diameters stacked on them. In the simplest version of the puzzle, all the disks are placed on one peg in order of decreasing size, so that the largest disk is on the bottom and the smallest at the top of the pile:



The goal is to move the entire tower to another peg, while obeying the following restrictions:

- Only one disk may be moved at a time.
- No disk may be placed on a smaller disk.

The main questions regarding this puzzle have been:

- What is the minimum number of moves required to transfer the entire tower from one peg to another?
- What is the move sequence that will achieve this?

The minimum number of moves for a tower of n disks was quickly shown to be $2^n - 1$, with the simple recursive solution as follows: Label the three pegs *start*, *goal*, and *temp*. To move n disks from the *start* peg to the *goal* peg via the *temp* peg,

- (1) If $n = 1$, move the one disk from *start* to *goal*.
- (2) If $n > 1$,
 - (i) Recursively move the top $n - 1$ disks from *start* to *temp* via *goal*.
 - (ii) Move the n -th disk from *start* to *goal*.
 - (iii) Recursively move the $n - 1$ disks from *temp* to *goal* via *start*.

This solution takes $2^n - 1$ moves:

- (1) If $n = 1$, $f(n) = 1 = 2^n - 1$
- (2) If $n > 1$, $f(n) = 2 * (2^{n-1} - 1) + 1 = 2^n - 2 + 1 = 2^n - 1$

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In 1908, Henry Ernest Dudeney published a book of puzzles which included a variation on the standard Tower of Hanoi [1]. Dudeney replaced the disks with wheels of cheese, but more importantly added a fourth peg, greatly complicating the problem. He described this problem as “The Reve’s Puzzle”. Then in 1939 the American Mathematical Monthly published a generalized Tower of Hanoi problem, asking for a solution for n disks on any number k of pegs [5]. Two years later, the same journal published a pair of solutions, by J. S. Frame and the puzzle’s proposer, B. M. Stewart [6]. Frame and Stewart offered algorithms for solving the problem in a minimum number of moves, but an editorial note pointed out that neither had proved their algorithm correct. This is where we stand today: no one has yet offered a proven solution to the generalized Tower of Hanoi problem, but most assume that the solutions of Frame and Stewart are correct.

Since Frame and Stewart’s solutions were later proven to be equivalent (see [3]), we will describe the simpler of the two, and refer to it as the “Frame-Stewart algorithm”. In this description we will use the following definition:

Definition 1.1. For $k \in \mathbb{N}$, $k \geq 3$, let $H_k : \mathbb{N} \rightarrow \mathbb{N}$ be a function which returns the minimum number of moves to solve the Tower of Hanoi problem for n disks on k pegs according to the Frame-Stewart algorithm. For example, $H_3(n) = 2^n - 1$ and $H_4(5) = 13$.

The Frame-Stewart algorithm is as follows: For n disks on k pegs, if $k = 3$, use the algorithm given above. If $n = 0$, obviously no moves are required. For $k > 3$, $n > 0$, choose an integer l satisfying $0 \leq l < n$ that minimizes the steps used in the following formula:

- Move the top l disks from the start peg to an intermediate peg; this can be accomplished in $H_k(l)$ moves (since the bottom $n - l$ disks do not interfere with movements at all).
- Move the bottom $n - l$ disks from the start peg to the goal peg using $H_{k-1}(n - l)$ moves. (Since one peg is occupied by a tower of smaller disks, it cannot be used in this stage.)
- Move the original l disks from the intermediate peg to the goal peg in $H_k(l)$ moves.

Therefore, the recursive formula for solutions to the Tower of Hanoi problem according to the Frame-Stewart algorithm is as follows:

$$H_k(n) = \begin{cases} 0 & \text{if } n = 0 \\ 2^n - 1 & \text{if } k = 3, n > 0 \\ \min_{0 \leq l < n} 2H_k(l) + H_{k-1}(n - l) & \text{if } k > 3, n > 0 \end{cases}$$

Remark 1.2. It is interesting to note that we could derive the results for 3-peg Tower of Hanoi from the Frame-Stewart algorithm using the degenerate case of $k = 2$. If, following our logical intuition, we set

$$H_2(0) = 0, \quad H_2(1) = 1, \quad H_2(n) = \infty \text{ for all } n > 1$$

then for $k = 3$, $n > 1$ we would have to set $l = n - 1$ (in order that our second term $H_2(n - l)$ be finite), and our formula becomes

$$H_3(n) = 2H_3(n - 1) + H_2(1) = 2H_3(n - 1) + 1$$

which was our original recursive solution for $k = 3$.

2. FRAME-STEWART DIFFERENCES

In this section we will prove some interesting numerical results regarding the solutions produced by the Frame-Stewart algorithm. Although these results are probably well-known to experts in the subject, the literature in the field is somewhat scattered. We hope that the clear and explicit presentation of these results and their proofs provided in this paper will be of interest to both newcomers and those familiar with the problem.

The main thrust of this section will be Theorem 2.3; the rest of the section will be dedicated to a proof of this Theorem.

(In what follows, all variables represent elements of $\mathbb{Z}_{\geq 0} := \{0, 1, 2, \dots\}$ unless stated otherwise.)

Remark 2.1. It may be useful to mention the following identity regarding binomial coefficients, as it will come up repeatedly in this paper.

$$\binom{a}{b} + \binom{a}{b+1} = \binom{a+1}{b+1}$$

A proof is provided in appendix A.

Definition 2.2. Let $D_k: \mathbb{N} \rightarrow \mathbb{N}$ be the function defined by $D_k(i) = H_k(i) - H_k(i-1)$. Thus, $D_k(i)$ represents the number of additional moves required by the Frame-Stewart algorithm as the number of disks is incremented from $n-1$ to n .

Theorem 2.3. *Let $n, k, x \in \mathbb{N}$, $n \geq 1$, $k \geq 3$. Then for the Tower of Hanoi problem on k pegs,*

$$\forall x \geq 0, \quad D_k(n) = 2^x \iff \binom{k-3+x}{k-2} < n \leq \binom{k-2+x}{k-2}$$

where $\binom{a}{b}$ is the usual binomial coefficient and $a < b \implies \binom{a}{b} = 0$. Equivalently, if we view D_k as a sequence $\{D_k(i)\}_{i \in \mathbb{N}}$, the sequence will consist of $\binom{k-3}{k-3}$ of the value 2^0 , followed by $\binom{k-3+1}{k-3}$ of the value 2^1 , $\binom{k-3+2}{k-3}$ of 2^2 , etc.

Example 2.4. For $k = 5$ the sequence $\{D_5\}$ is as follows:

$$\begin{aligned} \{D_5\} &= \left\{ \binom{5-3}{5-3} \text{ of } 2^0, \binom{5-3+1}{5-3} \text{ of } 2^1, \binom{5-3+2}{5-3} \text{ of } 2^2, \dots \right\} \\ &= \{1 \text{ of } 2^0, 3 \text{ of } 2^1, 6 \text{ of } 2^2, 10 \text{ of } 2^3, \dots\} \\ &= \{2^0, 2^1, 2^1, 2^1, 2^2, 2^2, 2^2, 2^2, 2^2, 2^2, 2^3, 2^3, \dots\} \\ &= \{1, 2, 2, 2, 4, 4, 4, 4, 4, 8, 8, 8, 8, 8, 8, 8, \dots\} \end{aligned}$$

Proof of Theorem 2.3. The proof of Theorem 2.3 will use double strong induction on the number of disks and the number of pegs. It will follow the following general outline:

- Prove that Theorem 2.3 holds for all n if $k = 3$.
- Prove that Theorem 2.3 holds for $n = 1$ for all $k > 3$.
- Prove that, assuming Theorem 2.3 holds for all $k < k_0$ and $n < n_0$, then Theorem 2.3 holds for $k = k_0$, $n = n_0$.

Base Case. *Theorem 2.3 holds for all n if $k = 3$.*

Proof. We need to show that $D_3(n) = 2^x \iff \binom{x}{1} < n \leq \binom{1+x}{1}$, i.e. $D_3(n) = 2^{n-1}$. But $H_3(n) = 2^n - 1$, so we have

$$\begin{aligned} D_3(n) &= H_3(n) - H_3(n-1) \\ &= 2^n - 1 - (2^{n-1} - 1) \\ &= 2^{n-1} \end{aligned} \quad \square$$

Base Case. *Theorem 2.3 holds for $n = 1$ for all $k > 3$.*

Proof. The statement of Theorem 2.3 reduces to showing that $D_k(1) = 1$. This is trivially true, since for all k it is easy to show $H_k(1) = 1$. \square

Inductive Step. *Assuming Theorem 2.3 for all $k' < k$ and $n' < n$, then Theorem 2.3 holds for k, n .*

In order to prove the inductive step, we will need the following definitions:

Definition 2.5. For the Tower of Hanoi problem on k pegs, we will call a number n of disks “perfect” if n can be represented as $n = \binom{k-2+x}{k-2}$ for some $x \geq 0$.

Definition 2.6. For k pegs and n disks, we will call a number ℓ with $0 \leq \ell < n$ an “optimal partition number” if

$$H_k(n) = 2H_k(\ell) + H_{k-1}(n - \ell)$$

Note that this means ℓ is an optimal partition number if and only if

$$\forall l \text{ such that } 0 \leq l < n, \quad 2H_k(\ell) + H_{k-1}(n - \ell) \leq 2H_k(l) + H_{k-1}(n - l)$$

In other words, ℓ is a partition number realizing the minimum number of moves using the Frame-Stewart algorithm.

The proof of the inductive step will consist of the following parts: Assuming that Theorem 2.3 holds for all numbers of disks less than n and for all numbers of pegs less than k ,

- We will determine some upper and lower bounds for the optimal partition number, ℓ , of n .
- We will show that within these bounds, all ℓ are optimal partition numbers.
- Using this information, we will be able to directly calculate $D_k(n)$ for the relevant n using the inductive hypothesis.

Lemma 2.7. *Assuming Theorem 2.3 up to but not including some number of disks n , then for n satisfying $\binom{k-3+x}{k-2} < n \leq \binom{k-2+x}{k-2}$, we can state the following about any optimal partition number ℓ :*

$$\binom{k-4+x}{k-2} \leq \ell \leq \binom{k-3+x}{k-2}$$

Note that our proof of the base case $n = 1$ implies that there does exist some $\binom{k-3+x}{k-2} < n$ satisfying Theorem 2.3, specifically $\binom{k-3+1}{k-2} = 1$.

Proof. If ℓ is an optimal partition number, then, by definition,

$$\forall l \text{ satisfying } 0 \leq l < n, \quad 2H_k(\ell) + H_{k-1}(n - \ell) \leq 2H_k(l) + H_{k-1}(n - l)$$

We will use a proof by contradiction: If an optimal partition number ℓ is outside of the aforementioned bounds, then there exists an l with $2H_k(l) + H_{k-1}(n - l) < 2H_k(\ell) + H_{k-1}(n - \ell)$.

Assume first that $\ell < \binom{k-4+x}{k-2}$. We will prove that $l = \ell + 1$ contradicts the above inequality. Since $l < n$, our inductive hypothesis, combined with the fact that $l \leq \binom{k-4+x}{k-2}$, implies that $D_k(l) \leq 2^{x-2}$. Additionally, $\ell < \binom{k-4+x}{k-2}$ and $n > \binom{k-3+x}{k-2}$ imply that $n - \ell > \binom{k-3+x}{k-2} - \binom{k-4+x}{k-2} = \binom{k-4+x}{k-3}$. Therefore, by our inductive hypothesis, $D_{k-1}(n - \ell) \geq 2^x$. Now, recalling that $D_k(n) = H_k(n) - H_k(n-1)$, we get

$$\begin{aligned} 2H_k(l) + H_{k-1}(n - l) &= (2H_k(\ell) + 2D_k(l)) + (H_{k-1}(n - \ell) - D_{k-1}(n - \ell)) \\ &\leq 2H_k(\ell) + 2(2^{x-2}) + H_{k-1}(n - \ell) - 2^x \\ &< 2H_k(\ell) + H_{k-1}(n - \ell) \end{aligned}$$

contradicting our definition of ℓ as an optimal partition number.

Next assume $\ell > \binom{k-3+x}{k-2}$ and compare it to $l = \ell - 1$. (Recall that we require $\ell < n$, so we can use the inductive hypothesis on ℓ and l .) Since $\ell > \binom{k-3+x}{k-2}$ and $n \leq \binom{k-2+x}{k-2}$, we have $n - \ell < \binom{k-2+x}{k-2} - \binom{k-3+x}{k-2} = \binom{k-3+x}{k-3}$ and $n - l \leq \binom{k-3+x}{k-3}$. Now the inductive hypothesis along with $\ell > \binom{k-3+x}{k-2}$ implies that $D_k(\ell) \geq 2^x$, and together with $n - l \leq \binom{k-3+x}{k-3}$ implies that $D_{k-1}(n - l) \leq 2^x$. Therefore

$$\begin{aligned} 2H_k(l) + H_{k-1}(n - l) &= (2H_k(\ell) - 2D_k(\ell)) + (H_{k-1}(n - \ell) + D_{k-1}(n - l)) \\ &\leq 2H_k(\ell) - 2(2^x) + H_{k-1}(n - \ell) + 2^x \\ &< 2H_k(\ell) + H_{k-1}(n - \ell) \end{aligned}$$

once again contradicting the definition of ℓ as an optimal partition number. \square

Lemma 2.8. *Assuming Theorem 2.3 up to but not including some number of disks n , then for n satisfying $\binom{k-3+x}{k-2} < n \leq \binom{k-2+x}{k-2}$ and any optimal partition number ℓ ,*

$$\binom{k-4+x}{k-3} \leq n - \ell \leq \binom{k-3+x}{k-3}$$

Proof. The proof is similar to that of Lemma 2.7: First assume $n - \ell < \binom{k-4+x}{k-3}$. Then $\ell > n - \binom{k-4+x}{k-3} > \binom{k-3+x}{k-2} - \binom{k-4+x}{k-3} = \binom{k-4+x}{k-2}$ so that $D_k(\ell) \geq 2^{x-1}$, and if we let $l = \ell - 1$ we get $D_{k-1}(n - l) \leq 2^{x-1}$. Then

$$\begin{aligned} 2H_k(l) + H_{k-1}(n - l) &= (2H_k(\ell) - 2D_k(\ell)) + (H_{k-1}(n - \ell) + D_{k-1}(n - l)) \\ &\leq 2H_k(\ell) - 2(2^{x-1}) + H_{k-1}(n - \ell) + 2^{x-1} \\ &< 2H_k(\ell) + H_{k-1}(n - \ell) \end{aligned}$$

so ℓ cannot be an optimal partition number. Therefore we know that $n - \ell \geq \binom{k-4+x}{k-3}$.

Now assume $n - \ell > \binom{k-3+x}{k-3}$. Then $D_{k-1}(n - \ell) \geq 2^{x+1}$, and $\ell < n - \binom{k-3+x}{k-3} \leq \binom{k-2+x}{k-2} - \binom{k-3+x}{k-3} = \binom{k-3+x}{k-2}$ so that if we let $l = \ell + 1$ then the inductive hypothesis implies $D_k(l) \leq 2^{x-1}$. Now we can calculate

$$\begin{aligned} 2H_k(l) + H_{k-1}(n - l) &= (2H_k(\ell) + 2D_k(l)) + (H_{k-1}(n - \ell) - D_{k-1}(n - \ell)) \\ &\leq 2H_k(\ell) + 2(2^{x-1}) + H_{k-1}(n - \ell) - 2^{x+1} \\ &< 2H_k(\ell) + H_{k-1}(n - \ell) \end{aligned}$$

contradicting the assumption that ℓ is an optimal partition number, so we conclude that $n - \ell \leq \binom{k-3+x}{k-3}$. \square

Remark 2.9. Note that for any perfect $n = \binom{k-2+x}{k-2}$, Lemma 2.7 states that all optimal partition numbers ℓ are $\leq \binom{k-3+x}{k-2}$, and Lemma 2.8 states that $n - \ell \leq \binom{k-3+x}{k-3} \iff \ell \geq \binom{k-3+x}{k-2}$, so we have proved that Theorem 2.3 also implies

$$n = \binom{k-2+x}{k-2} \implies \ell = \binom{k-3+x}{k-2}$$

In other words, when n is perfect the optimal partition number ℓ is both unique and perfect, as well as simple to calculate. We will return to this interesting point later.

Lemma 2.10. *Assuming Theorem 2.3 up to but not including some number of disks n , then for n satisfying $\binom{k-3+x}{k-2} < n \leq \binom{k-2+x}{k-2}$, if an integer ℓ satisfies the conditions stated in Lemmas 2.7 and 2.8, then ℓ is an optimal partition number.*

Proof. We will show that any choice of ℓ in the given range results in the same expression for $2H_k(\ell) + H_{k-1}(n - \ell)$. Since the definition of an optimal partition number states that the value of $2H_k(\ell) + H_{k-1}(n - \ell)$ is minimized, this implies that all ℓ in the given range are optimal.

Let $n = \binom{k-3+x}{k-2} + a$, where $0 < a \leq \binom{k-3+x}{k-3}$. Now set $\ell = \binom{k-4+x}{k-2} + b$ and $n - \ell = \binom{k-4+x}{k-3} + c$, where $a = b + c$. By Lemmas 2.7 and 2.8 we have $\binom{k-4+x}{k-2} \leq \ell \leq \binom{k-3+x}{k-2}$, and $\binom{k-4+x}{k-3} \leq n - \ell \leq \binom{k-3+x}{k-3}$.

By the bounds on ℓ (Lemma 2.7), and the inductive hypothesis we see that

$$\begin{aligned} H_k(\ell) &= H_k\left(\binom{k-4+x}{k-2}\right) + \left(\ell - \binom{k-4+x}{k-2}\right) 2^{x-1} \\ &= H_k(\ell - b) + (b)2^{x-1} \end{aligned}$$

while by the same argument the bounds on $n - \ell$ (Lemma 2.8) imply

$$\begin{aligned} H_{k-1}(n - \ell) &= H_{k-1}\left(\binom{k-4+x}{k-3}\right) + \left(n - \ell - \binom{k-4+x}{k-3}\right) 2^x \\ &= H_{k-1}(n - \ell - c) + (c)2^x \end{aligned}$$

Therefore,

$$\begin{aligned} 2H_k(\ell) + H_{k-1}(n - \ell) &= (2H_k(\ell - b) + (2b)2^{x-1}) + (H_{k-1}(n - \ell - c) + (c)2^x) \\ &= 2H_k(\ell - b) + H_{k-1}(n - \ell - c) + (b + c)2^x \\ &= 2H_k\left(\binom{k-4+x}{k-2}\right) + H_{k-1}\left(\binom{k-4+x}{k-3}\right) + a2^x \\ (*) \quad &= H_k(n - a) + a2^x \end{aligned}$$

This last line relies on the fact that by the inductive hypothesis we know that the (unique) optimal partition number for $n - a = \binom{k-3+x}{k-2}$ is $\ell = \binom{k-4+x}{k-2}$. Therefore $H_k(n - a) = 2H_k\left(\binom{k-4+x}{k-2}\right) + H_{k-1}\left(\binom{k-4+x}{k-3}\right)$.

Since we attain the same expression no matter what we choose for b and c (provided they fulfill the conditions stated in Lemmas 2.7 and 2.8), all such ℓ are optimal partition numbers. \square

Finally, (*) also completes the proof of Theorem 2.3 \square

3. CONSEQUENCES OF THEOREM 2.3

3.1. An iterative method for computing H_k . Now that we have more information about the values of H_k , we can calculate $H_k(n)$ without using a recursive formula:

Theorem 3.1. For $k \geq 3$, and $\binom{k-3+x}{k-2} < n \leq \binom{k-2+x}{k-2}$ for some $x \geq 0$,

$$H_k(n) = \sum_{i=0}^{x-1} \binom{k-3+i}{k-3} 2^i + \left(n - \binom{k-3+x}{k-3} \right) 2^x$$

Proof. This is a direct result of Theorem 2.3, since

$$\begin{aligned} H_k(n) &= H_k(0) + \sum_{i=1}^n D_k(i) \\ &= 0 + \sum_{i=0}^{x-1} \left(\sum_{j=\binom{k-3+i}{k-2}+1}^{\binom{k-2+i}{k-2}} (2^i) \right) + \sum_{i=\binom{k-3+x}{k-2}+1}^n (2^x) \\ &= \sum_{i=0}^{x-1} \left(\binom{k-2+i}{k-2} - \binom{k-3+i}{k-2} \right) 2^i + \left(n - \binom{k-3+x}{k-2} \right) 2^x \\ &= \sum_{i=0}^{x-1} \binom{k-3+i}{k-3} 2^i + \left(n - \binom{k-3+x}{k-3} \right) 2^x \quad \square \end{aligned}$$

3.2. Calculating ℓ directly. It would be useful to be able to calculate an optimal partition number ℓ directly from the number of disks n . In the following Theorem and Corollary, we provide a function that calculates ℓ directly from n for the 4-peg Tower of Hanoi.

Theorem 3.2. For the Tower of Hanoi puzzle on $k = 4$ pegs, for any number of disks n , if $\binom{x+1}{2} < n \leq \binom{x+2}{2}$, then

$$\ell = n - (x + 1)$$

is an optimal partition number.

Proof. Recall the bounds we defined for an optimal partition number ℓ in Lemmas 2.7 and 2.8: For n satisfying $\binom{k-3+x}{k-2} < n \leq \binom{k-2+x}{k-2}$, we require

$$\binom{k-4+x}{k-2} \leq \ell \leq \binom{k-3+x}{k-2} \quad \text{and} \quad \binom{k-4+x}{k-3} \leq n - \ell \leq \binom{k-3+x}{k-3}$$

For $k = 4$ this reduces to the following: For $\binom{x+1}{2} < n \leq \binom{x+2}{2}$,

$$\binom{x}{2} \leq \ell \leq \binom{x+1}{2} \quad \text{and} \quad \binom{x}{1} \leq n - \ell \leq \binom{x+1}{1}$$

The second condition is obviously satisfied by our choice of ℓ , since $n - \ell = x + 1$. The first condition is also satisfied:

$$\ell = n - (x + 1) \leq \binom{x+2}{2} - \binom{x+1}{1} = \binom{x+1}{2}$$

and

$$\begin{aligned} \ell &= n - (x + 1) > \binom{x+1}{2} - \binom{x+1}{1} \\ \implies \ell &= n - (x + 1) \geq \binom{x+1}{2} - \binom{x}{1} = \binom{x}{2} \end{aligned}$$

(The last equation is implied because we are dealing exclusively with integers.)

Since ℓ is within the bounds specified by Lemmas 2.7 and 2.8, Lemma 2.10 implies that ℓ is an optimal partition number. \square

Corollary 3.3. *For the Tower of Hanoi puzzle on $k = 4$ pegs, for any number of disks n ,*

$$\ell = n - \left\lfloor \sqrt{2n} + \frac{1}{2} \right\rfloor$$

is an optimal partition number.

Proof. First we create a function $g: \mathbb{N} \rightarrow \mathbb{N}$ which inverts the binomial function $f(x) = \binom{x}{2}$ in the following manner: $g(y) = x \iff \binom{x-1}{2} < y \leq \binom{x}{2}$. Then $g(y) - 2 = x \iff \binom{x+1}{2} < y \leq \binom{x+2}{2}$, and by Theorem 3.2, we can find an optimal partition number ℓ defined by

$$\ell = n - (x + 1) = n - ((g(n) - 2) + 1)$$

As proven in Appendix B, $g(y) = \lfloor \sqrt{2y} + \frac{3}{2} \rfloor$ inverts the binomial function in the manner described, and therefore

$$\ell = n - (g(n) - 2 + 1) = n - \left\lfloor \sqrt{2n} + \frac{3}{2} \right\rfloor - 1 = n - \left\lfloor \sqrt{2n} + \frac{1}{2} \right\rfloor$$

is an optimal partition number. \square

3.3. A simple recursive solution for perfect numbers. Recall from Remark 2.9 that when n is some perfect number $n = \binom{k-2+x}{k-2}$, we know that there is a unique optimal $\ell = \binom{k-3+x}{k-2}$, and therefore a unique optimal $n - \ell = \binom{k-3+x}{k-3}$. In particular, ℓ is perfect on k pegs, and $n - \ell$ on $k - 1$ pegs, so that each part of the recursive solution

$$H_k(n) = 2H_k(\ell) + H_{k-1}(n - \ell)$$

is also perfect and easy to calculate. In fact, the recursive solution bears a strong resemblance to the binomial equation that forms the basis for Pascal's triangle: $\binom{a+1}{b+1} = \binom{a}{b+1} + \binom{a}{b}$. In this case the equation is

$$H_k \left(\binom{a+1}{b+1} \right) = 2H_k \left(\binom{a}{b+1} \right) + H_{k-1} \left(\binom{a}{b} \right)$$

Using two variants of Pascal's triangle, we can quickly calculate $H_k(n)$ for any perfect number n of disks as follows:

Recall that the entries in Pascal's triangle, which we will denote P , have the form

$$P_{i,j} = \binom{i}{j} = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } i = 0, j > 0 \\ P_{i-1,j} + P_{i-1,j-1} & \text{if } i > 0, j > 0 \end{cases}$$

where $P_{i,j}$ is the entry in row i , column j of P , indexed from 0.

First we build a version of Pascal's triangle, P' , with the column indices offset by two, i.e. $P'_{i,j}$ is only defined on $i \geq 0, j \geq 2$ and $P'_{i,j} = P_{i,j-2}$ so that $P'_{i,2} = 1$ for all $i \geq 0$. This allows us to map x and k to perfect $n = \binom{x}{k-2}$ as follows: $P'_{x,k} = \binom{x}{k-2} = n$.

Next we build a second triangle P'' with the same column offset, but instead of Pascal's $P_{i,j} = P_{i-1,j} + P_{i-1,j-1}$ we set

$$P''_{i,j} = 2P''_{i-1,j} + P''_{i-1,j-1}$$

Then we can map x and k to $P''_{x,k} = H_k \left(\binom{x}{k-2} \right) = 2H_k \left(\binom{x-1}{k-2} \right) + H_{k-1} \left(\binom{x-1}{k-3} \right)$.

Now, using P' to calculate x based on n , and P'' to calculate $H_k(n)$ based on x , we can quickly and conveniently calculate $H_k(n)$ for any perfect number of disks n on k pegs:

P'	$k =$							P''	$k =$							
	2	3	4	5	6	7	8		2	3	4	5	6	7	8	
$x =$								$x =$								
0	1							0	1							
1	1	1						1	1	1						
2	1	2	1					2	1	3	1					
3	1	3	3	1				3	1	7	5	1				
4	1	4	6	4	1			4	1	15	17	7	1			
5	1	5	10	10	5	1			5	1	31	49	31	9	1	
6	1	6	15	20	15	6	1	6	1	63	129	111	49	11	1	

In order to find $H_k(n)$ for some perfect number n , first we look up n under column k in P' . Then we can discover x by looking at the index of the row containing n . Finally, we look up x, k in P'' to get $P''_{x,k} = H_k(n)$.

Example 3.4. If we want to look up $H_k(n)$ for the perfect number of disks $n = 15$ on $k = 4$ pegs, first we look for n in table P' under column $k = 4$. We discover that $P'_{6,4} = 15$, so $x = 6$. Then we look up $P''_{x,k} = P''_{6,4}$ to discover that $H_4(15) = 129$.

Example 3.5. To find $H_5(10)$, we first look up 10 in table P' under column $k = 5$. Then we see that $x = 5$, and looking up $P''_{5,5}$ shows that $H_5(10) = 31$.

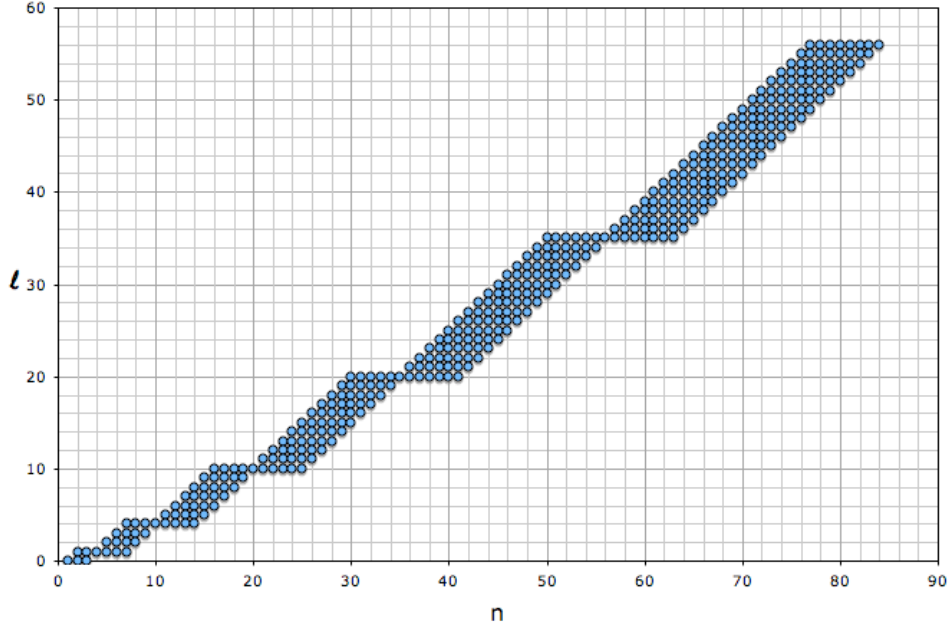


FIGURE 1. The set of all optimal partition numbers ℓ of n on 5 pegs

APPENDIX A. PROOF THAT $\binom{a}{b} + \binom{a}{b+1} = \binom{a+1}{b+1}$

Proof. By the definition of $\binom{a}{b} = \frac{a!}{(a-b)!b!}$, we have

$$\begin{aligned}
 \binom{a}{b} + \binom{a}{b+1} &= \frac{a!}{(a-b)!b!} + \frac{a!}{(a-b-1)!(b+1)!} \\
 &= \frac{a!(b+1)}{(a-b)!(b+1)!} + \frac{a!(a-b)}{(a-b)!(b+1)!} \\
 &= \frac{a!(a-b+b+1)}{(a-b)!(b+1)!} \\
 &= \frac{(a+1)!}{((a+1)-(b+1))!(b+1)!} \\
 &= \binom{a+1}{b+1}
 \end{aligned}$$

□

APPENDIX B. INVERSE OF $\binom{x}{2}$

Proposition B.1. *The function $g: \mathbb{N} \rightarrow \mathbb{N}$ defined by*

$$g(y) = \left\lfloor \sqrt{2y} + \frac{3}{2} \right\rfloor$$

fulfills the condition

$$g(y) = x \iff \binom{x-1}{2} < y \leq \binom{x}{2}$$

We will find the following lemmas useful in our proof:

Lemma B.2. For $a, b \in \mathbb{N}, a, b > 0$, $a < b \implies a \leq \lfloor \sqrt{ab} \rfloor$

Lemma B.3. For $a, b \in \mathbb{N}, a, b > 0$, $a \neq b \implies \sqrt{ab} < \frac{a+b}{2}$

Proof.

$$\begin{aligned} (a-b)^2 &> 0 \\ a^2 + b^2 &> 2ab \\ (a+b)^2 &> 4ab \\ \frac{(a+b)}{2} &> \sqrt{ab} \quad \square \end{aligned}$$

Lemma B.4. Let $f(x) = \frac{x(x-1)}{2} = \binom{x}{2}$ and $g(y) = \lfloor \sqrt{2y} + \frac{3}{2} \rfloor$. Then $\forall x \in \mathbb{N}$, $g(f(x)) = x$.

Proof. Examine $g(f(x))$:

$$\begin{aligned} g(f(x)) &= \left\lfloor \sqrt{2 \binom{x}{2}} + \frac{3}{2} \right\rfloor \\ &\geq \left\lfloor \sqrt{x(x-1)} + 1 \right\rfloor \\ &\geq (x-1) + 1 = x \quad (\text{by Lemma B.2}) \end{aligned}$$

and by Lemma B.3 we have

$$\begin{aligned} \sqrt{x(x-1)} &< \frac{x + (x-1)}{2} \\ \sqrt{x(x-1)} + \frac{3}{2} &< x - \frac{1}{2} + \frac{3}{2} = x + 1 \\ \left\lfloor \sqrt{x(x-1)} + \frac{3}{2} \right\rfloor &< x + 1 \end{aligned}$$

So $g(f(x)) = \left\lfloor \sqrt{x(x-1)} + \frac{3}{2} \right\rfloor$ is an integer greater than or equal to x and less than $x + 1$, i.e. $g(f(x)) = x$. \square

We can now complete the proof that $g(y) = \lfloor \sqrt{2y} + \frac{3}{2} \rfloor$ fulfills the necessary condition:

Proof. For $x, y \in \mathbb{N}$, if y satisfies $\binom{x-1}{2} = \frac{(x-2)(x-1)}{2} < y \leq \frac{(x-1)(x)}{2} = \binom{x}{2}$, then $g(y) = x$. First, we will prove that $y > \frac{(x-2)(x-1)}{2}$ implies $g(y) \geq x$: Since both y

and $\frac{(x-2)(x-1)}{2}$ are integers, we have

$$\begin{aligned} y &\geq \frac{(x-2)(x-1)}{2} + 1 \\ &= \frac{x^2 - 3x + 4}{2} \\ &> \frac{\left(x - \frac{3}{2}\right)^2}{2} \\ \sqrt{2y} + \frac{3}{2} &> x \\ \left\lfloor \sqrt{2y} + \frac{3}{2} \right\rfloor &\geq x \quad (\text{since } x \text{ is an integer}) \end{aligned}$$

Second, $y \leq \frac{(x-1)x}{2} = f(x)$ implies $g(y) \leq x$: Since $g(y)$ is an increasing function, and by Lemma B.4 we have $g(f(x)) = x$, then $y \leq f(x)$ implies $g(y) \leq g(f(x)) = x$. This completes our proof that

$$g(y) = x \iff \binom{x-1}{2} < y \leq \binom{x}{2} \quad \square$$

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