Knot homologies & low-dimensional topology

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What is a knot?

A knot is an equivalence class of smooth imbeddings

\[ f : S^1 \to \mathbb{R}^3, \]

where two imbeddings are equivalent if they can be connected by an isotopy.
Q: Why do we care about knots?

Q: Why do I, Eli Grigsby, a low-dimensional topologist, care about knots?

A: Fundamental for understanding 3- and 4-dimensional manifolds.
Q: Why 3- & 4-dimensional manifolds?

Dimensions <3: Very constrained—too easy

Dimensions 3, 4: Just right.

Dimensions >4: Too much freedom--translate questions to homotopy theory
Q: What do knots have to do with low-dimensional topology?

A: They can be used to construct new 3-manifolds from old.
Surgery on Knots

\[ K \subset \mathbb{R}^3 \]

\[ \mathbb{R}^3 - N(K) \]
Surgery on Knots

$\mathbb{R}^3 - N(K)$

$S^1 \times D^2$
Surgery on Knots

\[ \mathbb{R}^3 - N(K) \]

\[ \mathbb{R}^3 = [\mathbb{R}^3 - N(K)] \cup_{\mu=\partial D^2} [S^1 \times D^2] \]
Surgery on Knots

\[ \mathbb{R}^3 - N(K) \]

\[ \mathbb{R}^3(K) = [\mathbb{R}^3 - N(K)] \cup_{\gamma = \partial D^2} [S^1 \times D^2] \]

New 3-manifold!
Theorem (Lickorish-Wallace): Every closed, connected, oriented 3-manifold can be obtained by doing surgery on a link in $S^3$.

Moral: If we can understand knots (links), we can understand 3-manifolds.
How do we understand knots?

Complicated...
Idea: *Knot Invariants*

- Knot invariant
- Algebraic object
  - Number
  - Polynomial
  - Group
  - Graded group
  - Category
  - etc.

Prove invariance under choice of diagram: *Knot invariant*
Classical knot invariants

• Alexander polynomial
• Jones polynomial
• Determinant
• Signature
• Arf invariant
• Mod $p$ colorings
• Tristam-Levine signatures
• etc.
New knot invariants!

*Knot homology theories*

- *Powerful* (separate knots, encode topological information)
- *Computable* (a computer can do it)
Knot homology theories

$H(C_{i,j}, \partial)$

Homology of a Bi-graded chain complex
Knot homology theories

Knot Floer homology
(P. Ozsváth-Z. Szabó, Rasmussen)

Khovanov homology
(M. Khovanov)
Bi-graded chain complexes and Homology

\[ \partial : C_{i,j} \to C_{i,j-1} \text{ is a linear transformation} \]

\[ \partial \circ \partial = 0 \quad \Rightarrow \quad \text{Im}(\partial) \subseteq \text{Ker}(\partial) \quad \Rightarrow \quad H_{i,j} = \frac{\text{Ker}(\partial)}{\text{Im}(\partial)} \]

\[ C_{i,j} \text{ is a finite-dimensional vector space (over } F = \mathbb{Z}/2\mathbb{Z}) \]
Euler Characteristics: Knot homologies as “Categorifications”

\[ \widetilde{HFK}_{i,j}(K) \]

\[ K = 10_{145} \]

\[ T^{-2} + T^{-1} - 3 + T + T^2 \]
Knot Floer Homology categorifies the classical Alexander polynomial.

Khovanov Homology categorifies the classical Jones polynomial.
Khovanov Homology & Knot Floer Homology

• Inspired by ideas in physics
• “Categorify” classical knot invariants
• Share other formal similarities

Definitions are quite different...
Definition: Knot Floer Homology
(Manolescu-Ozsváth-Sarkar)
Definition: Knot Floer Homology

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Definition: Knot Floer Homology

(Manolescu-Ozsváth-Sarkar)

Grid Diagram for K
Grid Diagrams to Chain Complexes

Generators: 1-1 matchings of red/blue lines

Elements of $S_n$

Bijections of n-element set
Grid Diagrams to Chain Complexes

Differentials:
Count empty rectangles between generators differing by a transposition
Grid Diagrams to Chain Complexes

Not empty:
Contains an O
Grid Diagrams to Chain Complexes

Not empty: Contains an intersection point
Grid Diagrams to Chain Complexes

Bi-grading:
More complicated
(but not hard)
Grid Diagrams to Chain Complexes

Claim 1: $\partial^2 = 0$. 

*Not too hard.*

Claim 2: $\hat{HFK}(K)$ is independent of choice of grid diagram.

*Hard.*

$$H(C_{i,j}, \partial) = \hat{HFK}(K) \otimes V^{n-1}$$
Definition: Khovanov Homology

1

2

3

“0” res.

“1” res.
Definition: Khovanov Homology

(1,1,1) resolution
Definition: Khovanov Homology

$k$ circles

$V \otimes k = \text{Span}_F (v_+, v_-)$
Cube of Resolutions
Cube of Resolutions

\[ \partial \circ \partial = 0 \]
Cube of Resolutions

\[ \partial \circ \partial = 0 \]

Diagram-independent homology

\[ Kh(K) := H(C_{i,j}, \partial) \]
One (of many) Applications

Theorem (Ozsváth-Szabó):
\[ \hat{HFK}(K) \] detects the unknot.

Conjecture:
\[ Kh(K) \] detects the unknot.
One (of many) Applications

Theorem (Ozsváth-Szabó):
$\widehat{HFK}(K)$ detects the unknot.

Theorem (G-, Wehrli):
$\widehat{Kh}_n(K)$ detects the unknot ($n \geq 2$).
Connections?

Kh(K) and Double-branched cover (Ozsváth-Szabó)

\( \widehat{HF}(Y) \) : 3-manifold invariant, related to knot invariant

\( \widehat{Kh}(K) \) is related to \( \widehat{HF}(\Sigma(K)) \)

Double-branched cover of K
Connections?
Kh(K) and Double-branched cover

\[ \widetilde{Kh}(K) \]
Connections?
Kh(K) and Double-branched cover

\[ \widehat{HF}(\Sigma(K)) \]

Perturb the differential
Connections?
Triply-graded “superhomology”?  
(Dunfield, Gukov, Rasmussen)  

*Inspired by conjectured deep relationship in physics.*

\[ \mathcal{H}_{i,j,k}(K) \]

\[ \overset{\text{New Invariants}}{\downarrow} \quad \overset{\text{Other Known Invariants}}{\downarrow} \]

\[ \overset{HFK(K)}{\longrightarrow} \quad \overset{Kh(K)}{\longrightarrow} \]