TOWER OF HANOI RESEARCH PROJECT

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1. Introduction

My work this summer focused on problems relating to the Tower of Hanoi (TH) and its graph. For all of my work, I investigated theories and formulas regarding the Tower of Hanoi on three pegs (see image below), and attempted to expand them to Tower of Hanoi problems on four pegs.

2. Parity Restriction and its effects on the Tower of Hanoi Graph

2.1. Parity Restriction and the H_n^3 Graph.

The first part of my research was focused on a specific restriction on states in the Tower of Hanoi game, and I studied the effects of this restriction on the Tower of Hanoi graph.

Definition 2.1. According to [**?**], the H_n^k graph is a graph (k=number of pegs, n=number of disks) whose vertices, labelled by n-bit k-ary strings, correspond to legal configurations in the Tower of Hanoi on n disks (and k pegs) and where two vertices are adjacent if one can be obtained from the other by a legal move.

Objective 2.2. *To find a combination of state restrictions producing the shortest path between corner vertices of the H_n^3 graph, and therefore the least number of moves between perfect states in the H_n^3 game.*

Restriction 2.3. No odd-numbered disk may be placed directly upon another odd-numbered disk, and no even-numbered disk may be placed directly upon another even-numbered disk(In the above picture, this means that no white disk is allowed to touch another white disk, and no black disk is allowed to touch another black disk). Numerically, there cannot be an odd number of entries between two matching entries (listing the disks in order from largest to smallest, and writing them as the peg on which they are located, with the left-most peg being labeled 0).

Example 2.4. States 1001 and 1100 are legal, but states 1010 and 0202 are illegal.

Definition 2.5. An edge vertex is a vertex that lies on a geodesic between perfect states.

Open Question 2.6. For a given n, how many non-geodesic vertices in the H_n^3 graph satisfy Parity Restriction?

Remark 2.7. For relatively small n, we can solve for this value. The number of non-geodesic vertices satisfying Parity Restriction is equal to 3^n (the total number of vertices) minus $3(2^n - 1)$ (the number of geodesic vertices) minus the number of vertices not satisfying Parity Restriction. Therefore to solve for the number of non-geodesic vertices satisfying Parity Restricion, it is necessary to find the number of vertices in the H_n^3 graph not satisfying Parity Restriction. This value is easy to find for small n (it's 0 for n=1 and n=2 and 6 for n=3), and for greater values of n inclusion-exclusion counting can be used to determine its value, using the binary definition of Parity Restriction (There cannot be an odd number of entries between two matching entries). For values of n up to and including 5, there is no evident pattern in the numbers in the Lucas Correspondence (discussed in [**?**]) corresponding to the non-geodesic vertices in the H_n^3 graph satisfying Parity Restriction.

Open Question 2.8. Is there a restriction, or combination of restrictions on states in the H_n^3 graph that, when combined with the original restriction, would produce a H_n^3 graph which consisted of all edge vertices in the H_n^3 graph, and only edge vertices?

Remark 2.9. I then attempted to find a combination of restrictions on the H_n^4 game producing only edge vertices, which led to the following question.

Question 2.10. Do the states in the Frame-Stewart Algorithm all satisfy Parity restriction?

Definition 2.11. The Frame-Stewart Algorithm is the best method currently known for finding the shortest paths between perfect states in the H_n^4 graph. The Frame-Stewart Algorithm has three steps. The first step is moving the smallest n-ℓ-1 disks to a non-target peg. The second step is moving the remaining l+1 disks to the target peg using the three pegs (including the target peg) not containing the smallest n-l-1 disks. The third step is moving the smallest n-l-1 disks to the target peg.

Although the states in the middle part of the three-part Frame-Stewart Algorithm satisfied Parity restriction (since the middle stage is actually a three-peg Tower of Hanoi problem), states in the first and last part of the Frame-Stewart Algorithm do not satisfy Parity restriction. Therefore Parity Restriction is unable to provide a geodesic between perfect states of the H_n^4 graph.

3. Average Distances in the Tower of Hanoi Graph

My focus then shifted to finding average path lengths (representative of the average number of moves between states) in the Tower of Hanoi graph on three pegs, then, as before, progressing to four pegs.

3.1. Average Distances In the H_n^3 graph.

3.1.1. *Average Distance From a Perfect State to All Points in H^3_n.*

I began my research by studying the average distance from a corner vertex (representative of a perfect state in the H^3_n game) to any point (including the corner vertex itself) in the H^3_n graph. Since there exists a formula for this average distance (for all n), by studying the formula and its proof in [**?**], I gained insight into where to begin in my search for a formula for the average distance between all states in the Tower of Hanoi graph.

Definition 3.1. The average distance from the perfect state $\mathbf{i}$ to any state in H^k_n, denoted $A_n(\mathbf{i})$ is defined by

$$A_n(\mathbf{i}) := \sum_{a \in H^k_n} \frac{d(\mathbf{i}, a)}{|H^k_n|}$$

Theorem 3.2. [**?**]

$$A_n(\boldsymbol{i}) = \frac{2}{3}(2^n - 1)$$

Note: $\boldsymbol{i}$ *is a representative of the three corner vertices (equivalence defined later in Section* ***3.1.2****) and* $A_n(\boldsymbol{i})$ *is the average distance between a perfect state and all points in the* H^3_n *graph.*

3.1.2. *Average Distance Between Any Two Points in H^3_n.*

Next, I attempted to determine a formula for the average distance from any point in the H^3_n graph to any other point on the graph (including the original point).

Definition 3.3. The average distance associated to a graph Γ, denoted $A(\Gamma)$, is given by:

$$A(\Gamma) := \sum_{(a,b) \in V(\Gamma) \times V(\Gamma)} \frac{d(a,b)}{|V(\Gamma)|^2}$$

When I tried to find a formula for this value, I discovered equivalence classes.

Two vertices in the H^k_n graph are in the same equivalence class if there exists an element in the automorphism group of the graph (a group of maps that send one image of the graph to another, sending vertices to vertices, edges to edges, and preserving edge and vertex relations as well as graph shape and size) that sends one of the vertices to the other.

Example 3.4. The three perfect states in H^3_n are equivalent.

Theorem 3.5. *Let* Γ *be a general graph,* $V(\Gamma)$ *be the set of vertices of* Γ*,* $G(\Gamma)$ *be the group of automorphisms of the graph,* Γ*, and* Gv *denote the orbit of a vertex,*

$v \in V(\Gamma)$ under the action of $G(\Gamma)$. For the H_n^3 graph, for all v other than the three corner vertices, the size of Gv is six.

Proof. The vertices represent states, so a member of $G(\Gamma)$ represents essentially a re-ordering of piles of disks on the three pegs. For H_n^3, if all the disks are on the same peg (in a perfect state) at a vertex v, then the size of orbit Gv is three, since all disks can be on peg 0, 1, or 2. Otherwise, the orbit size of Gv is six, since when there are disks on multiple pegs there are three different stacks of pegs (even if one is an empty stack it's still unique from the other two stacks of disks), and there are three choices of stacks for peg 0, leaving two choices of stacks for peg 1, and one stack remaining to be placed on peg 2. □

Remark 3.6. For four pegs, there can up to 24 vertices in an orbit Gv (when there are three or four stacks of disks), but can also be 12 (only 2 stacks of disks) or 4 (1 stack of disks).

Additionally, $A(\Gamma)$ (the average distance between vertices) for a general graph, Γ, is equal to the sum of average distances from a specific v to all other points in Γ times Gv (orbit size for that specific v), for exactly one v from each subset Gv, all divided by $|V(\Gamma)|$.

Therefore, to find $\sum_{a,b \in H_n^3} d(a,b)$, it's necessary to find the distance from one member of each equivalence class to all points in the H_n^3 graph, multiply it by six (except the corner vertex representative, which is multiplied by three), and add the sum for each equivalence class. Using these equivalence classes to solve for A_n for some small values of n, I found

$$A_1 = \frac{2}{3}$$
$$A_2 = \frac{16}{9}$$
$$A_3 = \frac{946}{243}$$

Open Question 3.7. Is there a formula for A_n which works for all values of n?

3.2. Average Distances In the H_n^4 graph.

3.2.1. *Average Distance Between Any Two Points in H_n^4.*

First, I attempted to find a formula for the average distance between any point in the H_n^4 graph, and any other point on the graph (including the original point) for all n.

In a special case of Definition (**??**) where $\Gamma = H_n^k$,

$$A_n^4 = \sum_{a,b \in H_n^4} \frac{d(a,b)}{4^{2n}}$$

I began my search for a formula for $A_n^4(a,b)$ by solving for A_n^4 for small values of n and found

$$A_1^4 = \frac{3}{4}$$

$$A_2^4 = \frac{57}{32}$$

Open Question 3.8. Is there a formula for the average distance between any two points in the H_n^4 graph?

3.2.2. *Average Distance From a Perfect State to All Points in the H_n^4 Graph.*

$$A_n^4(\mathbf{i}) = \sum_{a \in H_n^3} \frac{d(\mathbf{i}, a)}{4^n}$$

I began the search for a formula by solving $A_n^4(\mathbf{i})$ for small values of n and found

$$A_1^4(\mathbf{i}) = \frac{3}{4}$$

$$A_2^4(\mathbf{i}) = \frac{33}{16}$$

$$A_3^4(\mathbf{i}) = \frac{231}{64}$$

Since the denominator of $A_n^4(\mathbf{i})$ is 4^n for all n, to find a formula for $A_n^4(\mathbf{i})$ all that's necessary is a formula for $\sum_{a \in H_n^4} d(\mathbf{i}, a)$. Solving for small values of n, I found

$$\sum_{a \in H_1^4} d(\mathbf{i}, a) = 3$$

$$\sum_{a \in H_2^4} d(\mathbf{i}, a) = 33$$

$$\sum_{a \in H_3^4} d(\mathbf{i}, a) = 231$$

Each of these numbers, as well as all values of $\sum_{a \in H_n^4} d(\mathbf{i}, a)$ are divisible by 3, since the three blocks which do not contain **i** are equivalent, and therefore have the same number of points the same distance away from **i**.

Open Question 3.9. Is there a formula for the average distance from a perfect state to all points in the H_n^4 graph?
Note: This and *Open Question 3.6* are extremely hard problems, since they contain, as a subproblem, the solution to the classical H_n^4 problem, the length of the geodesic between perfect states in H_n^4.

3.2.3. *Upper Bound on Average Distance from a Perfect State to All Points in the H_n^4 Graph.*

$$A_n^4 = \sum_{a \in H_2^4} \frac{d_n(\mathbf{i}, a)}{4^n}$$

Open Question 3.10. Let $D_n = \sum_{a \in H_2^4} d_n(\mathbf{i}, a)$ and FS_n =Frame-Stewart solution for n disks, the solution to the algorithm defined in Definition (**??**).
Is it possible to prove that $D_n > \sum_{a \in H_2^4} d_n(t, a)$ where t is any non-perfect state, and thereby prove that $D_n \leq D_{n-1} + 3(D_{n-1} + 4^{n-1} + \frac{4^{n-1} \times ((FS_n)-1)}{2})$, since D_{n-1} is the distance to the points in the same block as $\mathbf{i}$, $^{n-1}$ is the total number of moves to go from the transfer point in the same block as $\mathbf{i}$ to the transfer point in the target block, for all points not in the $\mathbf{i}$ block, and $\frac{4^{n-1} \times ((FS_n)-1)}{2}$ is and upper bound on how many moves it will take to move from $\mathbf{i}$ to the transfer point in the same block as $\mathbf{i}$, for each point in the other three blocks, and if it is possible to prove this, we know that the number of moves it takes to get from the transfer point in the target block to the target point in that block is less than the number of moves to get from the perfect state in that block to all other points therefore the number of moves to get from everywhere in the block to the perfect state can serve as an upper bound on the number of moves to get from everywhere in the block to the transfer point in that target block?

References

[PD] Poole, David (Dec. 1994). The Towers and Triangles of Professor Claus (or, Pascal Knows Hanoi) *Mathematics Magazine, Vol. 67, No. 5* 323-344