"Make-up" class Friday?

**Question:** Given a homology class \( A \in H_2(X+ \mathbb{R}^4) \) a 4-manifold, what is the minimal genus rep. ?

**Example:** Consider \( \mathbb{C}P^2 = \{ [x:y:z] \mid (x,y,z) \in \mathbb{C} \setminus \{0\}^3 \} \)

\([x:y:z] \sim [x':y':z'] \) if \( \exists \) the \( \mathbb{C} \setminus \{0\}^3 \) w/ \( (x,y,z) = (tx',ty',tz') \)

**Recall:** \( \mathbb{C}P^2 \) has a natural CW structure,

\[ S^2 = \mathbb{C}P^1 \]

\( S^2 \) is Point (0-cell) \( \{ [x:0:0] \mid x \in \mathbb{C} \setminus \{0\} \} \)

\( \{ [x:y:0] \mid y \neq 0 \} \)

\( \{ (\frac{x}{y},1,0) \mid \frac{x}{y} \in \mathbb{C} \} \)

\( \{ [x:y:z] \mid z \neq 0 \} \)

\( \{ (\frac{x}{z},\frac{y}{z},1) \mid \frac{x}{z} \in \mathbb{C} \} \)

\( \Rightarrow H_0(\mathbb{C}P^2) \cong H_2(\mathbb{C}P^2) \cong H_4(\mathbb{C}P^2) \cong \mathbb{Z} \)

We can find a lot of imbedded surfaces by studying nonsingular algebraic curves.

**Theorem (Genus-degree formula):** Let \( f(x,y,z) \) be an degree \( d \) homogeneous polynomial.

Then its zero set \( V_f \subseteq \mathbb{C}P^2 \)

is a complex algebraic curve (real smoothly-imbedded surface) of genus

\[ \frac{1}{2}(d-1)(d-2) \]

**Proof:** Adjunction formula for algebraic curves in 2D complex algebraic varieties.

**Principle:** The genus (Euler characteristic) of an algebraic
curve is determined by its homology class (degree).

A little bit of Bundle Theory

RECALL: Let $\mathbb{R}^k \to E^{n+k}$ be a real $k$-plane oriented bundle over an oriented closed, connected $n$-fold.

Define: The Euler class, $e(E) \in H^n(M^n)$ is the obstruction to choosing a non-vanishing section of $E$. (Recall: a section $s : E \to M$ is a smooth map $s : M \to E$ w/ $\pi \circ s = 1_M$)

Concretely:
1. Choose a "generic" section, $s : M \to E, s(M) \not\subseteq M$
2. Consider $s(M) \cap M \subseteq M$ the zero-section.

Check:
This class is independent of choice,

3. $s(M) \subseteq E$ are codim-$k$ submanifolds $M \subseteq E$.
   By transversality, $s(M) \cap M$ is a codim-$2k$ submanifold of $E$.
   \( \Rightarrow \) an oriented submfd. of $M$ of dim. $(n-k)$

4. Represents a homology class, $s_0 \in H_{n-k}(M)$.
5. $e(E) = \text{PD}_M(s_0)$.

Important special case: $k=n$ $s(M) \cap M$ is 0-dimensional.

Count points of intersection with sign $\to$ If $M^n$ is connected, then $H_0(M^n) \cong H^n(M^n) \cong \mathbb{Z}$.

Euler class is just this signed sum, as elt. of $\mathbb{Z}$, (cohomology class that assigns this signed sum to fundamental class of $M$).
Let $\mathbb{R}^2 \cong \mathbb{C} \rightarrow E^{n+2}$ be a complex line bundle over $M^n$, $M^n \cong \mathbb{R}$-dim. (oriented, connected)

**Defn.** The first Chern class, $c_1(E) \in H^2(M^n)$, is the obstruction to extending a trivialization on the 1-skeleton to a trivialization on the 2-skeleton.

**Concretely:**

1. Choose a handlebody decomposition for $M^n$ (by e.g., putting a Morse function on $M^n$).
   $$\Rightarrow$$ gives $M^n$ the homotopy type of a CW complex. (handles def. retract to cores).

2. Choose a trivialization on 0-skeleton, can extend to 1-skeleton (since $U(2)$ structure group, is connected).

3. Compare "canonical" trivialization on each 2-cell with a trivialization on its boundary (chosen trivialization on boundary undergoes a certain number of full rotations) $\Rightarrow$ Assign this integer to the 2-cell (Degree of map $\partial B^2 - S^1 \rightarrow S^1$).

**Claim:** This assignment represents a 2D cohomology class that is independent of choices.

**Proposition:** Let $\mathbb{C} \rightarrow E$ be a complex line bundle over a surface (oriented, connected, compact).

Then $c_1(E) = e(E)$ (Remark: $\mathbb{C}$ is naturally oriented).
(Recall: Noticed that in complex surfaces, the genus of a complex curve is determined by its homology class.) Say $M^2$ has genus $g$.

**Proof:** Noting that $M^2$ has a CW decomposition with one 0-handle $h_0$, $2g$ 1-handles $a_1, b_1, \ldots, a_g, b_g$, and one 2-handle $H_2$.

**Example:** $g = 2$

![Diagram](image)

We need only show that if $\langle c_1(E), [H_2] \rangle = \mathbb{R}$, then there is a section of $E$ whose algebraic intersection with $M^2$ is $\mathbb{R}$, (i.e., a vector field on $E$ with $k$ zeros of appropriate sign).

We have a trivialization on 1-skeleton that twists $k$ times with respect to the canonical trivialization of 2-cell:

**Trivialization on 1-skeleton** (boundary of single 2-cell) with twisting $K = 1$. (only real axis pictured).

**Canonical trivialization** of 2-cell (only real-axis vector pictured).

**Remark:** Trivialization on boundary defines a natural section $1 \in C_p \subseteq E_p$.
If \( k > 0 \), can fill in with a vector field (section) with \( k \) \underline{elliptic} zeroes

(modulus decreases as you move toward zero)

Can think of this as the gradient vector field where the local sources are \underline{minima} and sinks are local \underline{maxima}.

If \( k < 0 \), can fill in with a vector field (section) with \( k \) \underline{hyperbolic} zeroes

If \( k = 0 \), can fill in with a vector field \( 1 \) with no zeroes.

Convention: Elliptic / hyperbolic case corresponds to positive / negative intersections of push off with zero section.
Proof of Degree-Genus formula: Let $V_f$ be the zero locus of $f(x_0, y_1, z)$, a generic homogeneous polynomial of degree $d$. Let $[V_f] = d \in H_2(CP^2)$. Then $T(CP^2)|_V = T(V_f) \oplus N(V_f)$.

Whitney product formula $\Rightarrow c_1$ evaluated on any homology class is additive:

\[
\langle c_1(T(CP^2)), [V_f] \rangle = \langle c_1(TV_f), [V_f] \rangle + \langle c_1(N(V_f)), [V_f] \rangle.
\]

$3d$ (comes from the fact that Chern classes of tangent bundle of $CP^n$ well-understood:

\[
\langle c_1(TCP^n), [H] \rangle = n+1,
\]
and $[V_f] = d[H]$).

$\chi(V_f)$ (since $c_1(TV_f) = e(TV_f)$ and Euler class of tangent bundle evaluated on $V_f$ is Euler characteristic).

$2-2g(V_f)$ (since $c_1(N(V_f)) = e(N(V_f))$, and this is intersection of $V_f$ with a non-pushing). $d^2$.

So $g(V_f) = \frac{1}{2}(d^2 - 3d + 2) = \frac{1}{2}(d-2)(d-1)$. $\square$
Principle 1: In complex surfaces, the genus of an algebraic curve is determined by its homology class.

\((*)\) is a so-called "adjunction formula" in special cases of \(\mathbb{CP}^2\).

Principle 2: An algebraic curve is genus-minimizing in its homology class.

Thom conjecture (Now THEOREM, due to Kronheimer-Mrowka. Generalization to Morgan-Szabó-Taubes, Ozsváth-Szabó).

Let \(\Sigma\) be a smoothly-embedded connected surface representing the same homology class as an algebraic curve, \(C = \text{Vf}\).

Then, \(g(\Sigma) \geq g(C)(= \frac{1}{2}(d-1)(d-2))\) if \(f\) is degree \(d\).

We will be interested in "local version" more closely connected to classical knot theory.

Local Thom Conjecture (Now THEOREM, due to Kronheimer-Mrowka).

Let \(C\) be a nonsingular algebraic curve in \(\mathbb{C}^2\) (zero set of a generic polynomial in two variables), and \(B^4 \subseteq \mathbb{C}^2\) an imbedded 4-ball w/ \(\partial B^4 \cap C\).
\[ k \subseteq \partial B^4 = S^3. \]

Then \( g_4(k) = g(C \cap B^4). \)

Claim is that the portion of algebraic curve \( C \) inside \( B^4 \) is the genus-minimizer among all smoothly imbedded \( \mathbb{Z} \)-in same homology class as \( C \), i.e., among all bounded by \( k \).

Next time: State Topological Milnor conjecture (special case of local Thom conjecture for special class of knots: torus knots! )