(Principle of transversality in differential topology).

Last time: We noticed that the genus of an algebraic curve (smoothly-embedded surface) in a complex surface (real 4-D manifold w/ a complex structure) is determined by its homology class. Consequence of ADJUNCTION FORMULA (consequence of WHITNEY PRODUCT FORMULA for Chern classes ⇒ "additivity" of $c_1$).

Local Thom Conjecture: Let $C \subseteq \mathbb{C}^2$ be a nonsingular algebraic curve and $B^4 \subseteq \mathbb{C}^2$ an imbedded 4-ball w/ $\partial B^4 \cap C$ transversely (i.e., intersection is a smoothly-imbedded submanifold of $\mathbb{C}^2$ of codim 3). Then if $\partial B^4 \cap C = k$ is connected, $g_4(K) = g(C \cap B^4)$.

Special case of the above: Famous construction, advertised by Milnor (Singular points of Complex Hypersurface) which he attributes to Brauner.

Brauner construction: Let $f(z_1, \ldots, z_{n+1})$ be a complex polynomial, and let $V_f = \{ (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \mid f(z) = 0 \}$.

be an affine algebraic variety with isolated singularities.

Recall: A singular point $\hat{z}_0$ in this setting is one for which $\frac{\partial f}{\partial z_i} \bigg|_{\hat{z}_0} = 0 \quad \forall \ i$.

Singular points are isolated if each has an $\varepsilon$-ball in which no other singular points present.
Consider any $\bar{z}_0$ (isolated singularity) and let $\varepsilon > 0$ be such that $B^{2n+1}_\varepsilon(\bar{z}_0)$ contains no other singularities. $\varepsilon$-ball around $\bar{z}_0$.

Let $S^3_\varepsilon = \partial(B^{2n+2}_\varepsilon(\bar{z}_0)) = \{ \bar{z} \in \mathbb{C}^{n+1} \mid |\bar{z} - \bar{z}_0| = \varepsilon \}$.

We may assume (by changing $\varepsilon$ slightly, if necessary) that $V_f \cap S^3_\varepsilon$ transversely.

(i.e., $V_f \cap S^3_\varepsilon$ is a smoothly imbedded submanifold of $\mathbb{C}^{n+1}$ of codim$_{\mathbb{R}}(2 + 1) = 3$, i.e., dim$_{\mathbb{R}} = 2n - 1$)

Milnor Fibration Theorem: Consider $L_f = V_f \cap S^{2n+1}_\varepsilon \subseteq S^{2n+1}_\varepsilon$ (often called "link of singularity").

The argument function

$$\text{arg} f \circ \varepsilon = \frac{f}{|f|} : S^{2n+1}_\varepsilon \to S^1$$

fibers the complement of $L_f$.

I.e., $L_f$ is a "fibered link".

$$\text{arg}^{-1}(\theta) \to S^{2n+1}_\varepsilon \cap \mathbb{R}(L_f)$$

"Milnor fibre" $\to S^1$
Moreover, Milnor proves (Theorem 5.11) that the singularity at \( \overline{0} \) can be "smoothed" by perturbing \( V_f \) to \( V_{f,c} \) (for \( c \in \mathbb{C} \) sufficiently small \( c \in \mathbb{R} \))

\[
V_{f,c} = \left\{ \overline{z} \in \mathbb{C}^{n+1} \mid f(\overline{z}) = c \right\}.
\]

Without changing the isotopy class of \( L_f \subset S^{2n+1} \)

and \( V_{f,c} \cap B^{2n+2}_\varepsilon \) is diffeomorphic to \( \arg f^{-1}(\overline{0}) \). (in particular especially the Euler characteristic of \( \arg f(\overline{0}) \) matches that of \( V_{f,c}(B_\varepsilon) \).

Why is Brauner's construction / Milnor's theorem of interest to us?

When \( n = 1 \)

\( 0 \) \( \circ \) \( L_f \) is a link in \( S^3_\varepsilon \)

\( \circ \) \( V_f \) is a complex affine plane curve with an isolated singularity @ \( \overline{0} \)

\( \circ \) \( V_{f,c} \) is a complex affine plane curve which is nonsingular in \( B^{4}_\varepsilon(\overline{0}) \).

\[ \implies \text{in the setting of Local Thom conjecture:} \]

\( V_{f,c} \quad \circ \circ \circ \quad L_f \quad \text{matches} \quad X(L_f) \quad \text{Note:} \quad X(L_f) \quad \text{matches} \quad X(\text{push-in}) \]
Key example: Restrict to $n=1$ and consider
\[ f(z_1, z_2) = z_1^p + z_2^q \quad (p, q \in \mathbb{Z}^+) \]

Remark: $f$ has an isolated singularity at $z = 0$.
(Choose $\varepsilon \in \mathbb{R}$ small.)

Claim: $L_f = V_f \cap S^3$ is the $(p, q)$ torus link.

![Heegaard torus](image)

Flattened version:

\[ \begin{align*}
  \text{Heegaard torus} & \rightarrow S^3 \\
  \text{intersects} & \quad p \text{ times} \\
  \text{twist by} & \quad \frac{2\pi \varepsilon}{q} \\
  \text{p vertical} & \quad \text{strands} \\
  \text{of unity} & \quad \text{(at p\textsuperscript{th} roots of unity)}
\end{align*} \]

Braid representation: Closure of $(\sigma_1 \cdots \sigma_{q-1})^p \in B_q$.

Stack
\[ p \text{ copies} \]

\[ q \text{ strands} \]
Proof:

(Exercise) We are looking at all \((z_1, z_2) = (r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) \in \mathbb{C}^2\) such that:

1. \(z_1^p = -z_2^q\) (condition that \((z_1, z_2) \in V_f\))
2. \(r_1 + r_2 = \varepsilon\) (condition that \((z_1, z_2) \in S_\varepsilon^2\))

Right way to think of \(S_\varepsilon^2\): (Note \(r_2\) determined by \(r_1\)).

- \(z\) axis (where \(r_1 = \varepsilon, r_2 = 0\), param. by \(\theta_1\))
- "unit" circle in \(xy\)-plane (where \(r_1 = 0, r_2 = \varepsilon\), param. by \(\theta_2\))

In between, have tori \((\varepsilon' \times \varepsilon')\) of constant \(0 < r_1 < \varepsilon, 0 < r_2 < \varepsilon\) parameterized by \(\theta_1, \theta_2\).

Show:

1. \(f\) is imbedded on one of these tori of constant \((r_1, r_2)\).
2. For a fixed \(\theta_1\) slice, there are \(q\) solns. for \(\theta_2\)

Claim 2:

Seifert's algorithm yields a Seifert surface \(F_{pq}\) for \(T_{pq}\), with \(\{q\} 0\)-handles
\[
\chi(F_{pq}) = 1 - 2g(F_{pq}) = q - (q-1)p
\]
So
\[
g(F_{pq}) = \frac{1}{2} \left[ 1 - q + (q-1)p \right] = \frac{1}{2} (p-1)(q-1).
\]
Milnor proves (Theorem 5.11 in book): For $c \in C$ sufficiently close to 0, the "smoothing" of singularity $g$, i.e., the set

$$V_{f,c} = \{(z_1,z_2) \in C^2 \mid z_1^p + z_2^q = c, \|z_1\|^2 + \|z_2\|^2 \leq 3\}$$

is diffeomorphic to $F_{p,q}$.

**Note:** This is exactly the setting (special case) of Local Thom conjecture: an affine algebraic curve transversely intersecting $S^3$ in a link, $T_{p,q}$.

Say $T_{p,q}$ is a knot ($\text{gcd}(p,q) = 1$).

Adjunction inequality (principle 2) would tell us that

$$g(V_{f,c}) = \frac{1}{2}(p-1)(q-1) \leq g_4(T_{p,q})$$

We also know (by "push-in" construction)

$$g_4(T_{p,q}) \leq g(T_{p,q})$$

And we see a Seifert surface of genus $\frac{1}{2}(p-1)(q-1)$, so

$$g(T_{p,q}) \leq \frac{1}{2}(p-1)(q-1)$$

**Topological Milnor conjecture:** $g_4(T_{p,q}) = \frac{1}{2}(p-1)(q-1)$.

(We see from argument above that it follows from Local Thom conjecture and Milnor's work.)

**Next:** Show how to use Khovanov homology to get "local Thom..."