(Principle of transversality in differential topology).

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Last time: We noticed that the genus of a nonsingular algebraic curve (smoothly-imbedded surface) in a complex surface (real 4-D manifold w/ a complex structure) is determined by its homology class: consequence of ADJUNCTION FORMULA (consequence of WHITNEY PRODUCT FORMULA for Chern classes $\Rightarrow$ "additivity" of c_1)

Local Thom Conjecture: Let $C \subseteq \mathbb{C}^2$ be a nonsingular algebraic curve and $B^4 \subseteq \mathbb{C}^2$ an imbedded 4-ball w/ $\partial B^4 \cap C$ transversely (i.e., intersection is a smoothly-imbedded submanifold of $\mathbb{C}^2$ of $\operatorname{codim}_{\mathbb{R}} 3$ $= \dim_{\mathbb{R}} 1$) Then if $\partial B^4 \cap C = K$ is connected,

$$g_4(K) = g(C \cap B^4).$$

Special case of the above: Famous theorem/construction, advertised by Milnor (Singular points of Complex Hypersurfaces), which he attributes to Brauner.

Brauner construction:

Let $f(z_1, \dots, z_{n+1})$ be a complex polynomial, and let

$$V_f = \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} \mid f(\vec{z}) = 0\}.$$

be an affine algebraic variety with isolated singularities.

↳ Recall: A singular point $\vec{z}_0$ in this setting is one for which $\left.\frac{\partial f}{\partial z_i}\right|_{\vec{z}=\vec{z}_0} = 0 \quad \forall\, i$.

Singular points are isolated if each has an ε-ball in which no other singular points present.

Consider any $\vec{z}_0$ (isolated singularity) and let $\varepsilon > 0$ be such that $\underbrace{B_\varepsilon^{2n+2}(\vec{z}_0)}_{\varepsilon\text{-ball around } \vec{z}_0}$ contains no other singularities.

Let $S_\varepsilon^{2n+1} = \partial(B_\varepsilon^{2n+2}(\vec{z}_0))$
$= \{ \vec{z} \in \mathbb{C}^{n+1} \mid |\vec{z} - \vec{z}_0| = \varepsilon \}$.

We may assume (by changing ε slightly, if necess.) that

$$V_f \cap S_\varepsilon^3 \quad \underline{\text{transversely}}$$

(i.e., $V_f \cap S_\varepsilon^{2n+1}$ is a smoothly-imbedded submanifold of $\mathbb{C}^{n+1}$ of $\operatorname{codim}_{\mathbb{R}}(2+1) = 3$, i.e., $\dim_{\mathbb{R}} = 2n-1$)

<u>Milnor Fibration Theorem</u>: Consider $L_f = V_f \cap S_\varepsilon^{2n+1} \subseteq S_\varepsilon^{2n+1}$ (codim$_{\mathbb{R}}$ 2 in S_ε^{2n+1})

(often called "link of singularity").

The argument function

$$\arg_f := \frac{f}{|f|} : S_\varepsilon^{2n+1} \longrightarrow S^1$$

fibers the complement of L_f.

I.e., L_f is a "fibered link":

$$\begin{array}{ccc} \underbrace{\arg_f^{-1}(\theta_0)}_{\text{"Milnor fibre"}} & \longrightarrow & S_\varepsilon^{2n+1} - \nu(L_f) \\ & & \downarrow \\ & & S^1 \end{array}$$

(p. 2)

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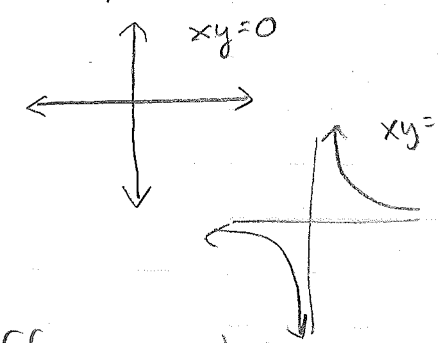
Moreover, Milnor proves (Theorem 5.11) that the singularity @ $\vec{z}_0$ can be "smoothed" by perturbing V_f to $V_{f,c}$: (for $c \in \mathbb{C}$ w/ sufficiently small $|c| \in \mathbb{R}$)

$$V_{f,c} = \left\{ \vec{z} \in \mathbb{C}^{n+1} \;\middle|\; f(\vec{z}) = c \right\}.$$

↳ Picture (real case):



without changing the isotopy class of

$$L_f \subseteq S^{2n+1}_{\varepsilon}.$$

and

$$V_{f,c} \cap B^{2n+2}_{\varepsilon} \text{ is diffeomorphic}$$

to $\arg_f^{-1}(\theta_0)$. (in particular: the Euler characteristic of $\arg_f^{-1}(\theta_0)$ matches that of $V_{f,c} \cap B_\varepsilon$).

Why is Brauner's construction / Milnor's Theorem of interest to us?

When $\underline{\underline{n=1}}$,
- L_f is a link in S^3_ε
- V_f is a complex affine plane curve with an isolated singularity @ $\vec{z}_0$.
- $V_{f,c}$ is a complex affine plane curve which is <u>nonsingular</u> in $B^4_\varepsilon(\vec{z}_0)$.

$\Longrightarrow$ in the setting of Local Thom conjecture:

Note: $\chi(L_f)$ matches $\chi(\text{push-in})$

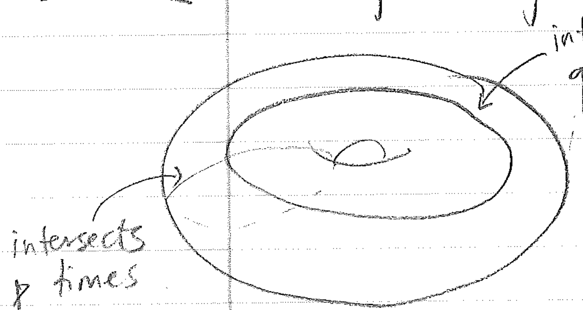
Key example: Restrict to $n=1$ and consider

$$f(z_1, z_2) = z_1^p + z_2^q \quad (p, q \in \mathbb{Z}^+).$$

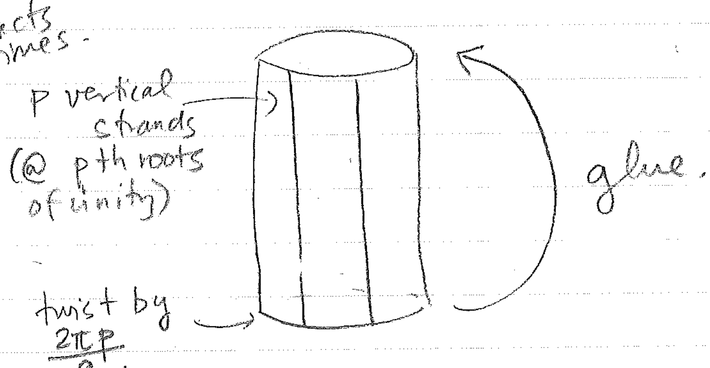
Remark: f has an isolated singularity @ $\vec{z} = \vec{0}$.

(Choose $\varepsilon \in \mathbb{R}$ small).

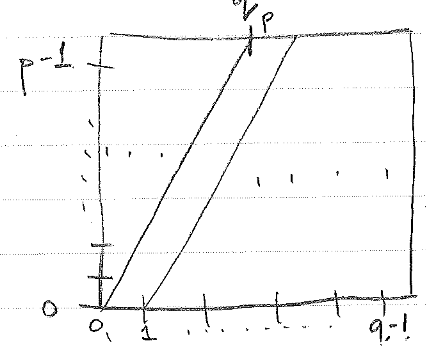
Claim 1: $L_f = V_f \cap S^3_\varepsilon$ is the (p,q) torus link:



Heegaard torus in S^3



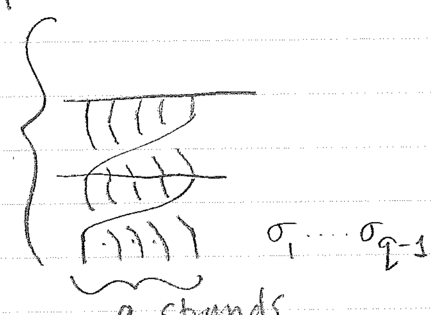
Flattened version:



slope $\frac{q}{p}$ curve

Braid representation: Closure of $(\sigma_1 \cdots \sigma_{q-1})^p \in B_q$.

Stack p copies



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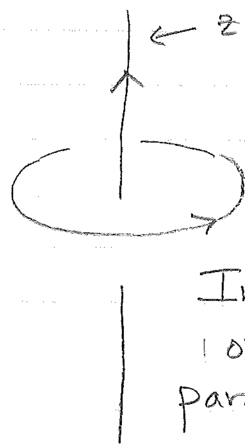
Proof: (Exercise) We are looking at all $(z_1, z_2) = (r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) \in \mathbb{C}^2$ such that

- $z_1^p = -z_2^q$ (condition that $(z_1, z_2) \in V_f$)
- $r_1^2 + r_2^2 = \varepsilon^2$ (condition that $(z_1, z_2) \in S^3_\varepsilon$)

Right way to think of S^3_ε: (Note → r_2 determined by r_1).



← z axis (where $r_1 = \varepsilon$, $r_2 = 0$, param. by θ_1)

← "unit" circle in xy-plane (where $r_1 = 0$, $r_2 = \varepsilon$, param. by θ_2)

In between, have tori ($S^1 \times S^1$) of constant $0 < r_1 < \varepsilon$, $0 < r_2 < \varepsilon$ parameterized by θ_1, θ_2.

Show:

① L_f is imbedded on one of these tori of constant (r_1, r_2).

② For each fixed θ_1 slice, there are q solns. for θ_2
" " θ_2 " , " " p " " θ_1.

Claim 2: Seifert's algorithm yields a Seifert surface, $F_{p,q}$, for $T_{p,q}$ with
$$\begin{cases} q & \text{0-handles} \\ (p-1)\cdot q & \text{1-handles} \end{cases}$$

$\Longrightarrow$ If $\gcd(p,q) = 1$ (i.e., $T_{p,q}$ is a knot - just one component).

$$\chi(F_{p,q}) = 1 - 2g(F_{p,q}) = q - (q-1)\cdot p$$

So
$$g(F_{p,q}) = \tfrac{1}{2}\left[1 - q + (q-1)\cdot p\right]$$
$$= \tfrac{1}{2}(p-1)(q-1).$$

Milnor proves (Theorem 5.11 in book):

For $c \in \mathbb{C}$ sufficiently close to 0, the "smoothing" of singularity, i.e., the set:

$$V_{f,c} = \left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; z_1^p + z_2^q = c, \; |z_1|^2 + |z_2|^2 \leq \varepsilon \right\}$$

is diffeomorphic to $F_{p,q}$.

<u>Note</u>: This is exactly the ~~setting~~ (special case) of Local Thom conjecture: an affine algebraic curve transversely intersecting S^3 in a *link*, $T_{p,q}$.

Say $T_{p,q}$ is a knot ($\gcd(p,q)=1$).

Adjunction <u>inequality</u> (principle 2) would tell us that

$$g(V_{f,c}) = \tfrac{1}{2}(p-1)(q-1) \leq g_4(T_{p,q}).$$

We also know (by "push-in" construction)

$$g_4(T_{p,q}) \leq g(T_{p,q})$$

And we see a Seifert surface of genus $\frac{1}{2}(p-1)(q-1)$, so

$$g(T_{p,q}) \leq \tfrac{1}{2}(p-1)(q-1).$$

<u>Topological Milnor conjecture</u>: $g_4(T_{p,q}) = \frac{1}{2}(p-1)(q-1)$.

(We see from argument above that it follows from Local Thom conjecture and Milnor's work.)

NEXT: SHOW HOW TO USE KHOVANOV HOMOLOGY TO GET "LOCAL THOM"