Recall: Want to prove topological Milnor conjecture: 
\[ g_4(T_{pq}) = \frac{1}{2} (p-1)(q-1). \]

Strategy:
1. Have obvious \( s \subseteq S^2 \) w/ \( \mathcal{C}(s) = T_{pq} \)
   
   \[ g(s) = \frac{1}{2} (p-1)(q-1). \]
   
   \( \Rightarrow g(T_{pq}) \leq \frac{1}{2} (p-1)(q-1) \)

2. "Push-in" construction \( \Rightarrow g_4(T_{pq}) \leq g(T_{pq}). \)

Need a way to bound \( g_4(T_{pq}) \) from below.

(via something like an adjunction inequality)

Rasmussen: Constructs a \( \mathbb{Z} \)-valued invariant of knots \( s(K) \) using Khovanov homology and further work of E.S. Lee:

- Shows \( s(K) \leq g_4(K) \)
- Shows \( s(T_{pq}) = \frac{1}{2} (p-1)(q-1) \)

Khovanov homology of links (over \( \mathbb{Q} \)).

Data: Link diagram

\( D(L) \)

\( c \)-dim cube

\( c \)-crossings

\( \mathcal{C}(L) \)

cube of resolutions

\( \mathcal{C}(s) \)

Functor

\( (1+1)-D \) = \( \{ \text{Ob: } \text{1-mflds. \& } \mathbb{Q} \} \)

\( \Rightarrow \{ \text{Ob: Vector-space/\( \mathbb{Q} \)} \}

\( \text{Mor: Orientable surfaces in } D \times I \)

\( \text{Mor: Linear maps} \)
Satisfies all desired functoriality properties.

\[ \emptyset \rightarrow \bigoplus \]  
(\text{empty object})

\[ \square \rightarrow \otimes \]  
(\text{ground field})

\[ \sqcup \rightarrow \otimes \]  
(\text{disjoint union})

\[ \otimes \rightarrow \text{Id}_{\text{V}, \text{V}} \]  
(\text{tensor product})

\[ \text{V} \otimes \text{W} \]  
(\text{"trivial" cobordism})

\[ \text{Id}_{\text{V}, \text{W}} \]  
(\text{identity map})

With the above properties in mind, we need only describe how the functor behaves on

\[ \emptyset \rightarrow \bigvee^{n} = \bigoplus_{i=1}^{n} \bigoplus \]  
(single circle)

\[ \bigvee^{m} \rightarrow \bigvee^{n} \]  
(two circles)

Saddle cobordisms

Merge

\[ \text{V} \otimes \text{V} \rightarrow \bigvee \rightarrow \bigvee \]  
(\text{"multiplication"})
Split $V \otimes V \rightarrow V \rightarrow V \otimes V$ "comultiplication."

$\Delta(1) = 1 \otimes x + x \otimes 1$
$\Delta(x) = x \otimes x.$

Cup $\rightarrow 2 : \mathbb{Q} \rightarrow V$ "unit"
$1 \mapsto 1.$

Cap $\varepsilon : V \rightarrow \mathbb{Q}$
$1 \mapsto 1,
 x \mapsto 1.$

"Hit with $(1+1)$-D TOFT" =

Replace vertices w/ vector spaces
Edges between adjacent vertices w/ linear maps.

Let $\bar{e} \in \{0,1,\infty\}^e$ # of crossings
$V_{\bar{e}} = \text{vector space at vertex } \bar{e}.$
E.g. $V_{00} = V_{\bar{0}}^2.$

Extra structure: 2 gradings.

(Co) Homological grading $\vdash$ Suppose $\bar{v} \in V_{\bar{e}}$

Let $\left| \bar{v} \right| = \# \text{ of } 1\text{'s in } \bar{e}.$
$\text{gr}(\bar{v}) = \left| \bar{v} \right| - n_\leftarrow \# \text{ of negative crossings in } \Delta(\bar{e}).$
"quantum" (internal): Let \( p(\vec{v}) \) be defined grading on "standard" basis vectors by:

\[
\begin{align*}
\circ p(v_{\pm}) &= \pm 1, \\
p(v_1 \otimes \ldots \otimes v_n) &= \sum_{i=1}^n p(v_i) \quad (\# +'s) \\
&\quad - \sum_{i=1}^n p(v_i) \quad (\# -'s)
\end{align*}
\]

(So, e.g., \( p(v_+ \otimes v_+) = 2 \), \( p(v_+ \otimes v_-) = 0 \), \( p(v_- \otimes v_-) = -2 \)).

Then \( q(\vec{v}) := p(\vec{v}) + |\vec{v}| + (n_+ - 2n_-) \)

(Circled terms are "normalization").

Theorem (Khovanov):

1. By sprinkling -1's appropriately on faces of cube, the above is a chain complex for any \( \partial \).

2. Boundary map \( \partial \) is sum of all maps \( m \otimes \Delta \), along edges "exiting" a vertex - edges oriented in direction of increasing homological grading.

(Check \( \partial^2 = 0 \))

2. \( \partial \) increases homol. grading by 1 and preserves \( q \)-grading.

2. The homology of \( C_*(\partial(L)) \) is an invariant \( \delta_b \) \( L \leq S^3 \), not \( \partial(L) \).

Notation: \( Kh^{i,j}(L) := \text{H}_i(C(\partial(L); j)) \)

\[ Kh(L) := \bigoplus_{ij \in \mathbb{Z}} Kh^{i,j}(L) \]
Idea of proof:

Motto (Michael Hutchings): (Morse/Floer homology setting)

1a) "Every oriented 1-manifold with boundary has an even number of boundary points, canceling with sign."

Each 1D flow line with boundary has broken flow lines (terms of \( S^2 \)) that cancel with sign.

2D Face of Kh cube analogue of 1D space of flow lines.

Check cancellation in a finite # of cases (configurations).

(1b) Check (easy).

2. One can associate to each Reidemeister move a homotopy equivalence of complexes.

Example (Reidemeister I):

\[
\begin{bmatrix}
\circ \\
\end{bmatrix} = \begin{bmatrix}
\circ \\
\end{bmatrix} \rightarrow \begin{bmatrix}
\circ \\
\end{bmatrix} + V
\]

Copy of \([\circ]\) Copy of \([\circ] \oplus V\).

Let \([\circ] + \) = subspace where trivial circle always marked w/ \( V + \)

Note that \([\circ] + \) is also a subcomplex (no arrows exiting).
Moreover, the map $\Delta$ in quotient complex:

\[
[\mathcal{C}] \xrightarrow{\Delta} [\mathcal{O}]
\]

is an isomorphism of complexes, hence acyclic (has zero homology). The LES on homology coming from SES:

\[
0 \rightarrow [\mathcal{O}]_+ \rightarrow [\mathcal{C}] \rightarrow \left\{ \left[\mathcal{C} \rightarrow [\mathcal{O}] \right] \right\} \rightarrow 0
\]

\[\xrightarrow{\text{subcomplex}} \quad \xrightarrow{\text{total}} \quad \xrightarrow{\text{quotient}} \]

implies that $H_*(\left[\mathcal{O}\right]_+) \cong H_*(\left[\mathcal{C}\right])$

\[\uparrow \text{note: Isomorphic to } H_*[\mathcal{C}] \text{ with } (h,q)\text{-grading shift of } (1,1) \]

Check gradings also agree, (once normalized).

Next time: Lee's "deformation" of Khovanov homology Rasmussen's filtration + surface cobordisms.