p. 1

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Recall: Want to prove topological Milnor conjecture:
$$g_4(T_{p,q}) = \tfrac{1}{2}(p-1)(q-1).$$

Strategy: ① Have obvious $\Sigma \subseteq S^3$ w/ $\partial(\Sigma) = T_{p,q}$, $g(\Sigma) = \frac{1}{2}(p-1)(q-1)$.
$$\Rightarrow g(T_{p,q}) \le \tfrac{1}{2}(p-1)(q-1)$$
② "Push-in" construction $\Rightarrow g_4(T_{p,q}) \le g(T_{p,q})$.

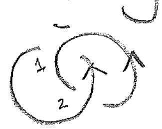
Need: ③ A way to bound $g_4(T_{p,q})$ from below. (via something like an adjunction inequality)

Rasmussen: Constructs a $\mathbb{Z}$-valued invariant of knots, $s(K)$ using Khovanov homology and further work of E.S. Lee:

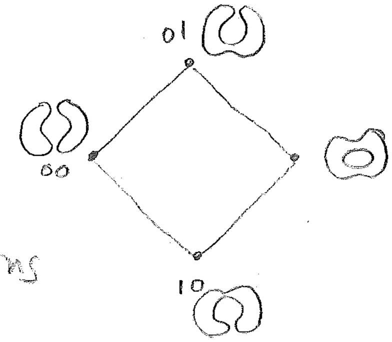
- Shows $s(K) \le g_4(K)$
- Shows $s(T_{p,q}) = \frac{1}{2}(p-1)(q-1)$

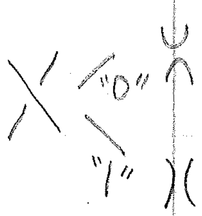
Khovanov homology of links (over $\mathbb{Q}$).

Data: (oriented) Link diagram



$\mathcal{D}(L)$ $\xrightarrow{\text{cube of resolutions}}$





c crossings

c-dim cube
vertices $\leftrightarrow \{0,1\}^c$

$\Longrightarrow$ (1+1)-D TQFT = Functor $\left\{\begin{array}{l}\underline{\text{Category}} \\ \text{Ob: } 1\text{-mfds} \subseteq D^2 \\ \text{Mor} = \text{Orientable surfaces in } D^2 \times I\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\underline{\text{Category}} \\ \text{Ob} = \text{Vector spaces}/\mathbb{Q} \\ \text{Mor} = \text{Linear maps}\end{array}\right\}$ (Grading-preserving)

Satisfies all desired functoriality properties.

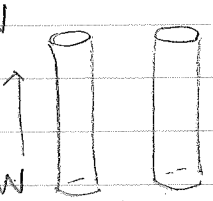
$\emptyset \longmapsto \mathbb{Q}$

(empty object) (ground field)

$\amalg \longmapsto \otimes$

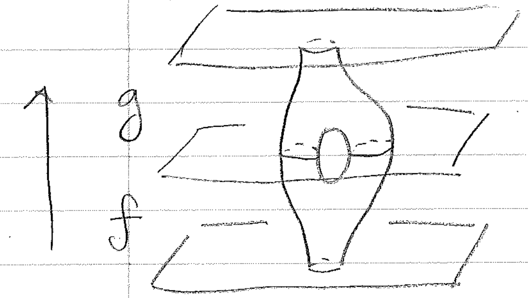
(disjoint union (tensor product).

$V\otimes W$



$V\otimes W$ $\longmapsto$ $\mathrm{Id}_{V\otimes W}$

("trivial" cobordism) (identity map).



g

f $\longmapsto$ $g\circ f$

Also: Associativity of cobordisms, independent of Morse decomp.

With the above properties in mind, we need only describe how the functor behaves on

$\bigcirc \longmapsto V = \mathbb{Q} \oplus \mathbb{Q}$

(single circle) $= \mathrm{Span}_{\mathbb{Q}}\{v_+, v_-\}$

$= \mathrm{Span}_{\mathbb{Q}}\{1, x\} \cong \mathbb{Q}[x]/x^2.$

($\bigcirc\ \bigcirc \longmapsto$ $V^{\otimes 2} \cong \mathbb{Q}[x]/x^2 \otimes \mathbb{Q}[x]/x^2$

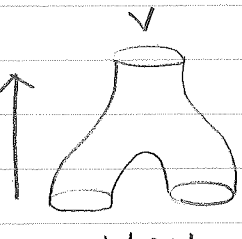
(two circles)

$\cong \mathbb{Q}[x_1, x_2]/(x_1^2, x_2^2).$

Saddle cobordisms

V

Merge



$V\otimes V$

$\longmapsto V\otimes V \longrightarrow V$ "multiplication"

$1\otimes 1 \longrightarrow 1$ m

$1\otimes x \quad x\otimes 1 \longrightarrow x$

$x\otimes x$

(p. 2)

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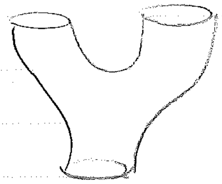
Split $\uparrow$ 

$V\otimes V$

V

$\longmapsto$ $V \longrightarrow V\otimes V$ "comultiplicat." Δ.

$1 \xrightarrow{1\otimes 1} x\otimes 1$, $1 \longrightarrow 1\otimes x$

$x \longrightarrow x\otimes x$.

$\Delta(1) = 1\otimes x + x\otimes 1$
$\Delta(x) = x\otimes x.$

Cup $\uparrow$ 

V

$\mathbb{Q}$

$\longmapsto$ $\iota : \mathbb{Q} \to V$ "unit"
$1 \mapsto 1.$

Cap $\uparrow$ 

$\mathbb{Q}$

V

$\longmapsto$ $\varepsilon : V \longrightarrow \mathbb{Q}$
$1 \longmapsto 0$
$x \longmapsto 1.$

"Hit with (1+1)-D TQFT" = Replace vertices w/ vector spaces
Edges between adjacent vertices w/ linear maps.

Hopf Link example

$$\begin{tikzcd} & V_{01} \arrow[dr, "\Delta"] & \\ V^{\otimes 2}_{00} \arrow[ur, "m"] \arrow[dr, "m"'] & \oplus & V^{\otimes 2}_{11} \\ & V_{10} \arrow[ur, "\Delta"'] & \end{tikzcd}$$

Let $\vec{\varepsilon} \in \{0,1\}^c$ ← # of crossings
$V_{\vec{\varepsilon}}$ = vector space at vertex $\vec{\varepsilon}$.
E.g.: $V_{00} = V^{\otimes 2}$.

Extra structure: 2 gradings.

(Co) Homological grading: Suppose $\vec{v} \in V_{\vec{\varepsilon}}$

Let $|\vec{v}| :=$ # of 1's in $\vec{\varepsilon}$.

$gr(\vec{v}) := |\vec{v}| - n_-$ ← # of negative crossings in $\mathcal{D}(L)$.

"quantum" (internal) grading: Let $p(\vec{v})$ be defined on "standard" basis vectors by:

- $p(v_\pm) := \pm 1$.
- $p(v_1 \otimes \dots \otimes v_n) := \sum_{i=1}^{n} p(v_i)$ $\left(\begin{matrix}\# +\text{'s} -\\ \# -\text{'s}\end{matrix}\right)$

(So, e.g., $p(v_+ \otimes v_+) = 2$, $p(v_+ \otimes v_-) = 0$, $p(v_- \otimes v_-) = -2$).

Then $q(\vec{v}) := p(\vec{v}) + |\vec{v}| + n_+ - 2n_-$

(Circled terms are "normalization").

Theorem (Khovanov):

(1a) By sprinkling -1's appropriately on faces of cube, the above is a chain complex for any $\mathcal{D}$. (Boundary map ∂ is sum of all maps, m & Δ, along edges "exiting" a vertex - edges oriented in direction of increasing homological grading). (Check: $\partial^2 = 0$)

e.g. as described in Bar-Natan.

link diagram

(1b) ∂ increases homol. grading by 1 and preserves q-grading.

(2) The homology of $C_*(\mathcal{D}(L))$ is an invariant of $L \subseteq S^3$, not $\mathcal{D}(L)$.

Notation: $Kh^{i,j}(L) := H_i(C(\mathcal{D}(L); j))$

i: homol. grading; j: quantum grading.

Bigraded

$$Kh(L) := \bigoplus_{i,j \in \mathbb{Z}} Kh^{i,j}(L)$$

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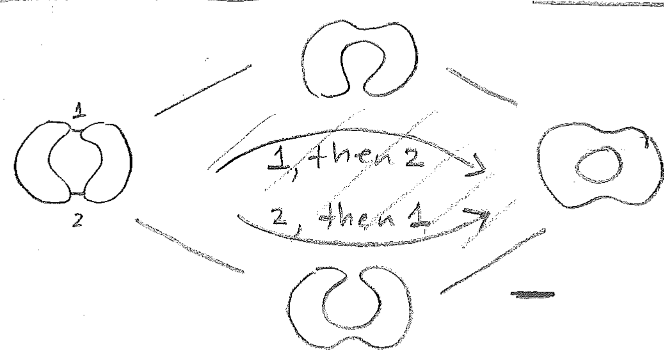
Idea of proof:

Motto (Michael Hutchings): (Morse/Floer homology setting)

(1a) "Every oriented 1-manifold with boundary has an even number of boundary points, canceling with sign."

$\Longrightarrow$ Check: Each 1D flow line with boundary has broken flow lines (terms of ∂^2) that cancel with sign.

2D-Face of Kh cube analogue of 1D space of flow lines.

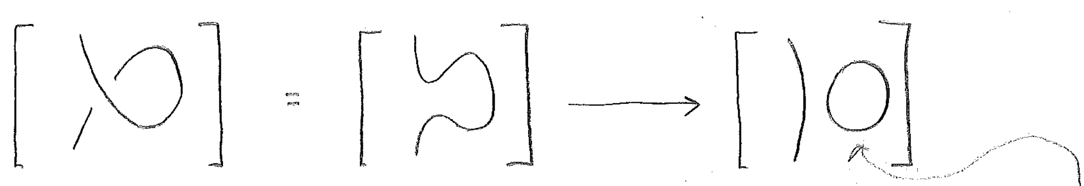


Check cancelation in a finite # of cases (configurations)

(1b) Check (easy).

(2) One can associate to each Reidemeister move a homotopy equivalence of complexes.

Example (Reidemeister I):

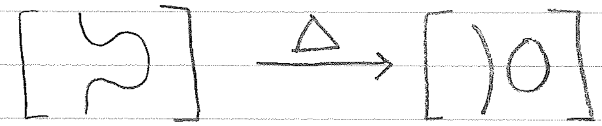


Copy of [ )] Copy of [ )] $\otimes$ V.

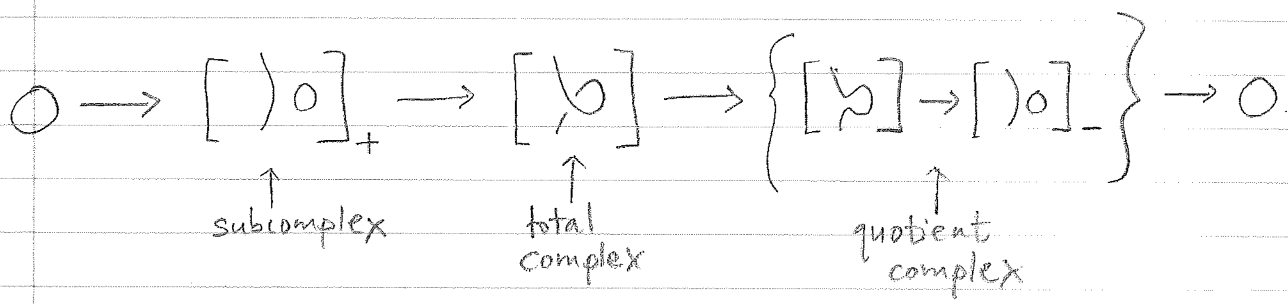
Let $[\,)\bigcirc\,]_+$ = subspace where trivial circle always marked w/ v_+

Note that $[\,)\bigcirc\,]_+$ is also a subcomplex (no arrows exiting).

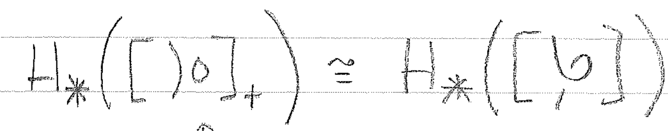
Moreover, the map Δ in quotient complex:



is an isomorphism of complexes, hence acyclic (has zero homology). The LES on homology coming from SES:



implies that



note: Isomorphic to $H_*[\ )\ ]$ with (h,q)-grading shift of $(1,1)$. $\square$

Check: gradings also agree (once normalized).

Next time: Lee's "deformation" of Khovanov homology
Rasmussen's filtration + surface cobordisms.