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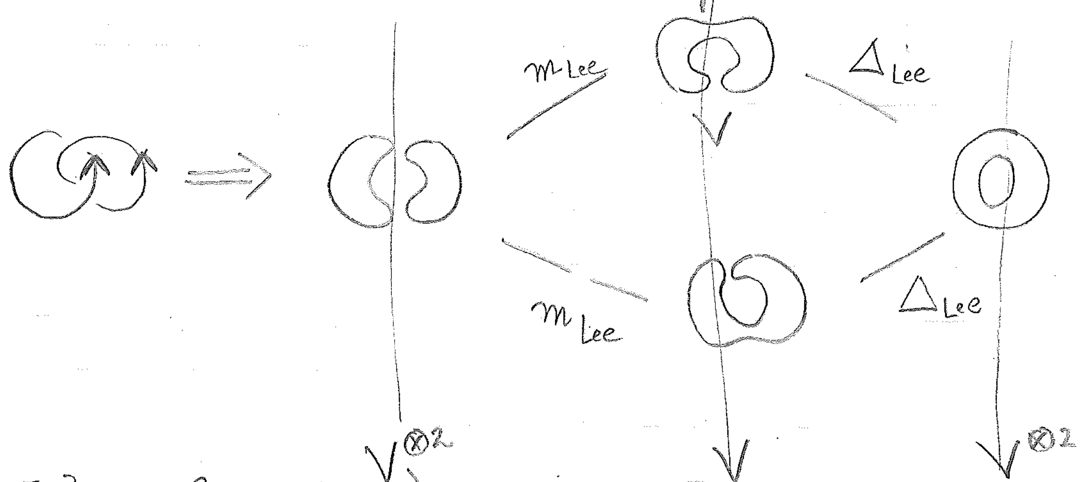
Last time: Introduced Khovanov TQFT, indicated how to prove Kh homology is an invariant of $L \subseteq S^3$.

Today: Describe Lee's TQFT, a <u>deformation</u> of Kh construction.

<u>Lee's TQFT</u>:
- Same objects
- "Deformed" morphisms.

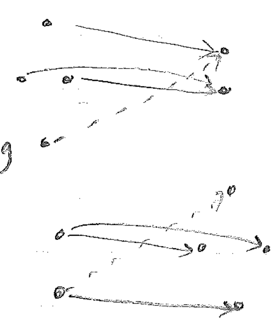


$(\eta_{Lee} = \eta_{Kh},\ \varepsilon_{Lee} = \varepsilon_{Kh})$.

∂_{Lee} is sum of all m_{Lee}, Δ_{Lee} exiting a vertex

$$\begin{cases} m_{Lee}(v_+ \otimes v_+) = v_+ \\ m_{Lee}(v_- \otimes v_+) = m_{Lee}(v_- \otimes v_+) = v_- \\ m_{Lee}(v_- \otimes v_-) = \boxed{v_+} \\ \Delta_{Lee}(v_+) = v_+ \otimes v_- + v_- \otimes v_+ \\ \Delta_{Lee}(v_-) = v_- \otimes v_- + \boxed{v_+ \otimes v_+} \end{cases}$$

increases q grading by 4.



<u>Remark</u>: Lee's differential, ∂_{Lee}, is <u>not</u> q-grading preserving. But it <u>is</u> monotonic in q:

$$q(\partial_{Lee} \vec{v}) \geq q(\vec{v})$$

$$\underset{\substack{\uparrow \\ \text{preserve } q}}{\partial_{Lee} = \partial_0} + \underset{\substack{\uparrow \\ \text{increase } q \text{ by } 4.}}{\partial_4} \qquad (\text{Note: } \partial_0 = \partial_{Kh})$$

Theorem (Lee): (1) $(\partial_{Lee})^2 = 0$ (Lee's deformation is a complex).
(2) $\mathcal{H}_{Lee} := H_*(\mathcal{C}_{Lee}, \partial_{Lee})$ is an invariant of L.

Important point: $(\mathcal{C}_{Lee}, \partial_{Lee})$ has the structure of a (finite-length, i.e. bounded) $\mathbb{Z}$-filtered complex whose associated graded complex is Khovanov's complex $(\mathcal{C}_{Kh}, \partial_{Kh})$

Defn: Let $(\mathcal{C}, \partial)$ be a chain complex. A (finite-length) filtration on $(\mathcal{C}, \partial)$ is a sequence of nested subcomplexes:

$$(\mathcal{C}, \partial) = \mathcal{F}_n \supseteq \quad \ldots \supseteq \mathcal{F}_{N-1} \supseteq \mathcal{F}_N = 0$$

In case of Lee's deformation:

$$\mathcal{F}_m := \left(\mathrm{Span}_{\mathbb{Q}}\{\vec{v} \in \mathcal{C} \mid q(\vec{v}) \geq m\},\ \partial_{Lee}|_{\mathcal{F}_m}\right)$$

restricted differential

(Since ∂_{Lee} non-decreasing, these are all subcomplexes
Since $\mathcal{C}_{Lee}$ is finite-dimensional, filtration is finite-length

Defn: If $(\mathcal{C}, \partial) = \mathcal{F}_n \supseteq \ldots \supseteq \mathcal{F}_N = 0$ is a $\mathbb{Z}$-filtered complex as above, then the associated graded complex is defined as:

$$\left(\bigoplus_{m \in \mathbb{Z}} \mathcal{F}_m / \mathcal{F}_{m+1},\ \partial_0\right)$$

map induced by ∂ on quotient.

Associated graded complex splits into graded pieces indexed by $\mathbb{Z}$.

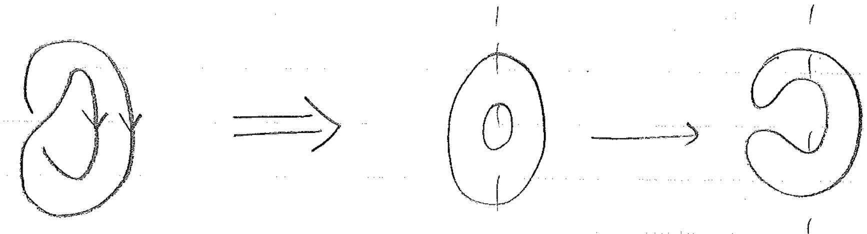
(p. 2)

Claim: For any link diagram $\mathcal{D}$

$$(\mathcal{C}_{Kh}(\mathcal{D}), \partial_{Kh}) \cong \text{assoc. gr.}\,(\mathcal{C}_{Lee}(\mathcal{D}), \partial_{Lee}).$$

viewed as a filtered complex.

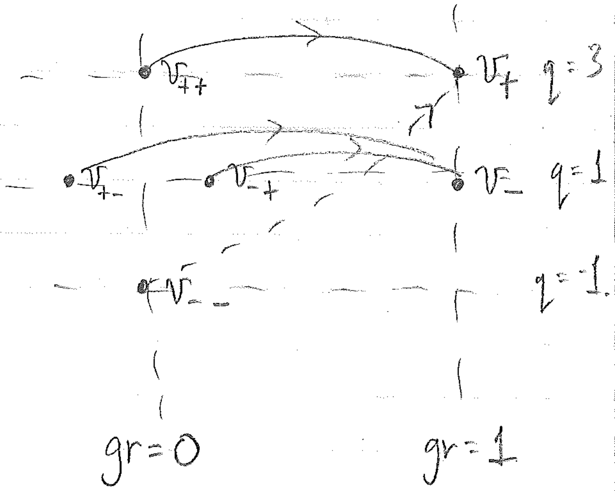
Example:



$n_+ = 1,\ n_- = 0$

$gr(\vec{v}) = |\vec{v}| - n_-$

$q(\vec{v}) = p(\vec{v}) + |\vec{v}| + n_+ - 2n_-$



$\mathcal{F}_4 = \mathcal{F}_5 = \ldots = 0$

$\mathcal{F}_2 = \mathcal{F}_3 = \text{Span}\{v_{++}, v_+\}$

$\mathcal{F}_0 = \mathcal{F}_1 = \text{Span}\{\quad, v_{+-}, v_{-+}, v_-\}.$

$\mathcal{C}_{Lee} = \ldots = \mathcal{F}_{-2} = \mathcal{F}_{-1} = \text{Span}\{\quad, v_{--}\}.$

Associated graded of filtered complex:

$$\mathcal{F}_3/\mathcal{F}_4 \qquad v_{++} \longrightarrow v_+$$

$$\oplus$$

$$\mathcal{F}_1/\mathcal{F}_2 \qquad v_{+-} \longrightarrow v_-, \quad v_{-+} \longrightarrow v_-$$

$$\oplus$$

$$\mathcal{F}_{-1}/\mathcal{F}_0 \qquad v_{--}$$

(no boundary since $v_+ \in \mathcal{F}_3 \subseteq \mathcal{F}_0$)

$\Longrightarrow$ The Lee filtration yields a <u>spectral sequence</u> whose E^1 term is Khovanov homology and whose E^∞ term is Lee homology.

(We will return to this point later.)

Key: The q-grading on Khovanov homology induces an "s"-grading on E^∞ page (Lee homology) (This is what Rasmussen uses to get g_4 bound)

<u>Theorem (Rasmussen)</u>: The filtered chain homotopy type of $(\mathcal{C}_{Lee}, \partial_{Lee})$ (whose filtration is given, as above, by q-grading) is an invariant of $L \subseteq S^3$ (doesn't depend on $\mathcal{D}(L)$).

Defn: Two filtered chain complexes $\mathcal{C}, \mathcal{C}'$ are filtered chain homotopy equivalent if $\exists$ <u>filtered</u> chain maps

$$\mathcal{C} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{C}' \qquad \begin{pmatrix} f(\mathcal{F}_i) \subseteq \mathcal{F}_i' \\ g(\mathcal{F}_j') \subseteq \mathcal{F}_j \end{pmatrix}$$

and <u>filtered</u> homotopies $\mathcal{C} \circlearrowleft H$, $\mathcal{C}' \circlearrowleft H'$

such that
$$g \circ f = id + (\partial H + H\partial)$$
$$f \circ g = id + (\partial' H' + H'\partial').$$

<u>Justifies</u>:

Definition
$$s_{min}(L) = \min\{ s(x) \mid x \in \mathcal{H}_{Lee}(L),\ x \neq 0 \}.$$
$$s_{max}(L) = \max\{ s(x) \mid x \in \mathcal{H}_{Lee}(L),\ x \neq 0 \}$$

(We will give primer on algebra of spectral sequences, induced grading, next time)

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Structure of Lee homology for $L \subseteq S^3$ a link

Theorem: (Lee) Let $L \subseteq S^3$ be an l-component link. Then $\mathcal{H}_{Lee}(L) := H_*(\mathcal{C}_{Lee}, \partial_{Lee}) \cong (\mathbb{Q} \oplus \mathbb{Q})^{\otimes l}$.

Moreover, there is a "canonical" identification
$$\left\{ \begin{array}{c} \text{orientations} \\ \text{of } L \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{basis elements} \\ \text{of } \mathcal{H}_{Lee} \end{array} \right\}.$$

Translation: ① Each orientation gives rise to a nonzero element in $\mathcal{H}_{Lee}$

② The set of these elements is linearly independ. and span $\mathcal{H}_{Lee}$

③ We know Reidemeister moves identify orientations; the corresponding (filtered) chain maps also identify the associated generators (up to mult. by nonzero $a \in \mathbb{Q}$).

For a knot K, $\mathcal{H}_{Lee}(K) \cong \mathbb{Q} \oplus \mathbb{Q}$.

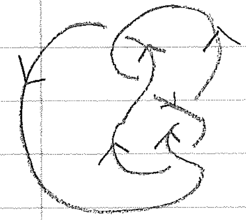
Proposition: (Rasmussen) $s_{min}(K) + 2 = s_{max}(K)$.

$\Longrightarrow$ Defn: $s(K) = s_{min}(K) + 1 = s_{max}(K) - 1$.

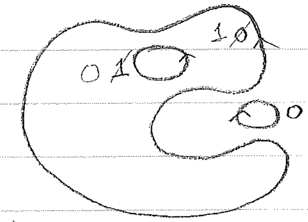
TODAY: Describe generators associated to two orientations of a knot; convince you that they are linearly independent in homology.

Lee's "canonical" generators (for a knot K).

Start: Orientation "o" on K
(on a diagram of K)



Oriented resolution



(one of vertices of cube).

Let $\left.\begin{array}{l} a = v_- + v_+ \\ b = v_- - v_+ . \end{array}\right\}$ Alternative $\mathbb{Q}$-basis for $V = \mathrm{Span}_{\mathbb{Q}}\{v_+, v_-\}$.

Mark a circle, C, of oriented resolution with a, b as follows:

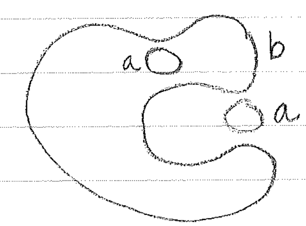
① Determine parity of # circles "enclosing" C (separating C from ∞)

② Add $\left\{\begin{array}{c} 1 \\ 0 \end{array}\right\}$ if orientation on C induced by "o" on K is $\left\{\begin{array}{c} \mathrm{CCW} \\ \mathrm{CW} \end{array}\right\}$.

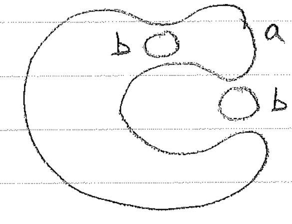
Mark with $\left\{\begin{array}{c} a \\ b \end{array}\right\}$ if result of ① & ② is $\left\{\begin{array}{c} 0 \\ 1 \end{array}\right\}$

$\Rightarrow$ Canonical generator "s_o" associated to orientation "o".

Example above:

$s_o =$ 

Easy Remark: If $\bar{o}$ is the opposite orientation, $s_{\bar{o}}$ is the generator with all labels switched: $a \longleftrightarrow b$.

$s_{\bar{o}} =$ 

(In particular, s_o & $s_{\bar{o}}$ are at same vertex, so $gr(s_o) = gr(s_{\bar{o}})$.)