2/24/14

Recall: • Lee constructed a deformation of Khovanov complex whose associated graded complex is Khovanov complex
($\Rightarrow$ Lee complex is filtered by q grading
$\Rightarrow$ get a well-defined "s" grading on Lee homology of a knot in S^3, $\mathcal{H}_{Lee}(K)$.)

TODAY
• Lee proves: $\mathcal{H}_{Lee}(K) \cong \mathbb{Q} \oplus \mathbb{Q}$.

• Rasmussen proves: As an s-graded vector space, $gr(\mathcal{H}_{Lee}(K)) \cong \mathbb{Q}_{s_{min}(K)} \oplus \mathbb{Q}_{s_{max}(K)}$

where $s_{max}(K) = s_{min}(K) + 2$.

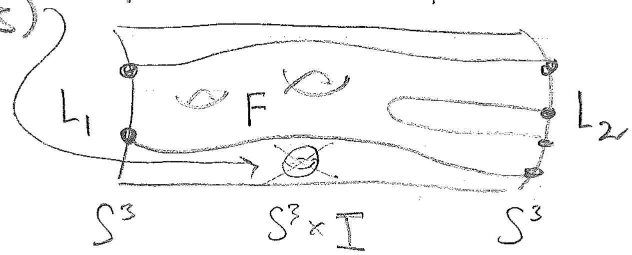
$\Rightarrow$ Defines $s(K) = s_{min}(K) + 1 = s_{max}(K) - 1$.

NEXT TIME
(Then shows $|s(K)| \leq 2g_4(K)$).

MORAL REASON why s(K) has nice properties

① $$\text{Basis elts. of } \mathcal{H}_{Lee}(L \subseteq S^3) \,/\, \text{nonzero scalar} \in \mathbb{Q} \xleftrightarrow{1:1} \text{Orientations on } L \subseteq S^3$$

② A 4-D cobordism $(S^3, L_1) \xrightarrow{(S^3 \times I, F)} (S^3, L_2)$ (with no closed components) induces a filtered map of degree determined by $\chi(F)$.

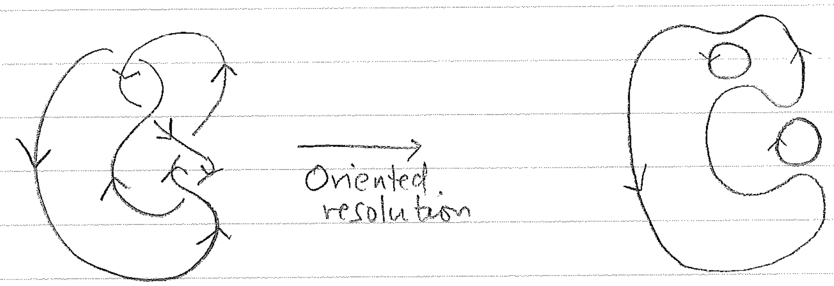


This map sends a generator associated to an orientation to a linear comb. (w/ all nonzero coefficients) of generators associated to <u>compatible</u> orientations.

($\exists$ an orientation on cobordism F restricting to given orientations on L_1, L_2).

Orientation "o" on $\mathcal{D}(L)$ $\rightsquigarrow$ Generator $s_o \in C_{Lee}(\mathcal{D}(L))$

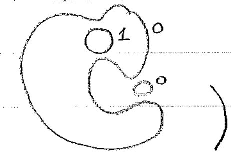
<u>diagram</u>



Define $a = v_- + v_+$
$b = v_- - v_+$.

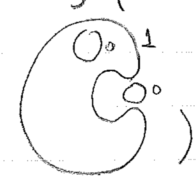
Mark circles of oriented resolution w/ $\left\{\begin{matrix} a \\ b \end{matrix}\right\}$ according to rule:

① Compute # (mod 2) of circles w/in which given circle is nested. (In the above,



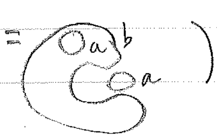
)

② Add $\left\{\begin{matrix} 1 \\ 0 \end{matrix}\right\}$ (again, mod 2) if orientation coming from resolution is $\left\{\begin{matrix} \text{CCW} \\ \text{CW} \end{matrix}\right\}$. (In the above,



)

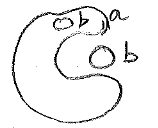
③ $\left\{\begin{matrix} 0 \\ 1 \end{matrix}\right\} \to \left\{\begin{matrix} a \\ b \end{matrix}\right\}$. (In the above, $s_o =$



)

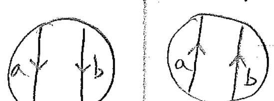
2/24/13

Remarks: ① If o is an orientation, and $\bar{o}$ is the opposite orientation, $s_{\bar{o}}$ is the generator obtained from s_o by replacing $\begin{Bmatrix} a \\ b \end{Bmatrix}$ w/ $\begin{Bmatrix} b \\ a \end{Bmatrix}$.

(In the above, $s_{\bar{o}}$ = )

We also see that $gr(s_o) = gr(s_{\bar{o}})$, & is determined by classical information.

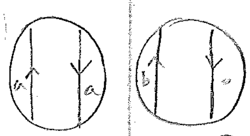
② Lemma (2.4 in Rasmussen): Adjacent strands in oriented resolution are either



- oriented in the same direction w/ opposite labels

OR

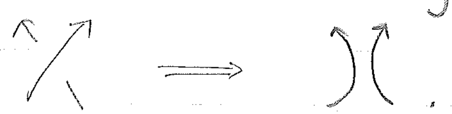
- oriented oppositely w/ the same label.



Proof: Check 3 cases: Same circle, 2 nested circles, 2 unnested circles.

Corollary: If two circles of s_o share a crossing, they have different labels.

Proof: After taking oriented resolution of a crossing, the two strands are oriented in the same direction locally.



NOW RESTRICT TO CASE THAT $L = K$ is a knot. (case of interest to us), so there are two distinguished generators of $\mathcal{C}_{Lee}(\mathcal{D}(K))$ corresponding to the two orientations, o and $\bar{o}$.

Proposition (Lee):

① $s_o, s_{\bar{o}}$ are cycles in $\mathcal{C}_{Lee}(\mathcal{D}(K))$ (hence represent homology classes)

② $[s_o], [s_{\bar{o}}]$ are linearly independent in $\mathcal{H}_{Lee}(K)$.

③ $[s_o], [s_{\bar{o}}]$ span $\mathcal{H}_{Lee}(K)$.

Proof:

① With respect to "new" basis $\{a, b\}$ for V (and corresponding $\underbrace{\text{"standard" basis}}$ for $V^{\otimes k}$, $k \geq 1$)

indexed by 2^k possible "binary" k-tuples (in $\{a, b\}$)

Ex. ($k=3$)

$V^{\otimes 3} = \mathrm{Span}_{\mathbb{Q}}\{a\otimes a\otimes a, a\otimes a\otimes b, a\otimes b\otimes a, b\otimes a\otimes a, a\otimes b\otimes b, b\otimes a\otimes b, b\otimes b\otimes a, b\otimes b\otimes b\}$

the Lee merge/split maps are (check):

Merge
$$\begin{cases} m^{Lee}(a\otimes a) = 2a \\ m^{Lee}(a\otimes b) = m^{Lee}(b\otimes a) = 0 \quad (*) \\ m^{Lee}(b\otimes b) = -2b. \end{cases}$$

Split
$$\begin{cases} \Delta^{Lee}(a) = a\otimes a \\ \Delta^{Lee}(b) = b\otimes b \end{cases}$$

By Lemma 2.4 in Rasmussen, any crossings on s_o will connect circles with opposite labels $\Longrightarrow$ ∂^{Lee} is a sum of merge maps (since split maps involve crossing between a circle and itself $\rightarrow$ same label).

But $(*)$ (and Lemma 2.4) $\Longrightarrow$ $m^{Lee}(s_o) = 0$ for all of these.

$\Longrightarrow$ $\partial^{Lee}(s_o) = 0$ and $\partial^{Lee}(s_{\bar{o}}) = 0$

$\Longrightarrow$ s_o and $s_{\bar{o}}$ are cycles $\rightarrow$ represent elts. of Lee homology. □

2/24/14

② a) Lee proves: $[s_o] \neq 0$ and $[s_{\bar{o}}] \neq 0$.
b) Rasmussen proves: $s_o + s_{\bar{o}}$ and $s_o - s_{\bar{o}}$ have different "mod 4" q-gradings ($\Rightarrow [s_o] \neq c \cdot [s_{\bar{o}}]$)

a) "Hodge theory":

The basis $\{a, b\}$ for V (and induced standard basis on $V^{\otimes k}$) gives rise to a nondegenerate bilinear pairing (inner product) for which standard basis is an orthonormal basis.

(Think: Standard inner product w/ respect to this basis $\Rightarrow \langle v, w \rangle := v^T w$).

Allows us to define a "codifferential", δ^{Lee}, adjoint to differential, ∂^{Lee}:

$$\langle \delta^{Lee} v, w \rangle := \langle v, \partial^{Lee} w \rangle \quad \forall \text{ pairs } v, w.$$

(Check: $\delta^{Lee} \circ \delta^{Lee} = 0$)

Lee's Claim: In this situation, (δ adjoint to ∂ with respect to nondegenerate pairing on $\mathcal{C}$).

$$H_*(\mathcal{C}, \partial) \cong \ker(\partial) \cap \ker(\delta).$$

Won't prove this general claim here. Instead, will prove weaker statement:

Easier claim: If $x \overset{\neq 0}{\in} \ker(\partial) \cap \ker(\delta)$, then $[x] \neq 0$ in $H_*(\mathcal{C}, \partial)$.

Proof: Since $x \in \ker \partial$, $[x] \in H_*(\mathcal{C}, \partial)$.
To see $[x] \neq 0$, just need $x \notin \operatorname{Im} \partial$.

Suppose $x = \partial y$ for $y \in \mathcal{C}$.
Then $\langle x, x \rangle = \langle x, \partial y \rangle = \langle \delta x, y \rangle = 0$, since $\delta x = 0$.

But $\langle \cdot, \cdot \rangle$ is a nondegenerate pairing, so $x = 0 \in \mathcal{C}$.
Since we assumed $x \neq 0 \in \mathcal{C}$, we conclude
$x \notin \operatorname{Im} \partial \implies [x] \neq 0 \in H_*(\mathcal{C}, \partial)$.

$\square$

Apply the above to $s_o, s_{\bar{o}}$ to see
$[s_o], [s_{\bar{o}}] \neq 0$.

(Direct computation:

merge/splits going backwards in cube.

$$\begin{cases} m^*: a \otimes a \to a \\ \quad a \otimes b, \ b \otimes a \to 0 \\ \quad b \otimes b \to b \\ \Delta^*: a \to 2a \otimes a \\ \quad b \to -2b \otimes b. \end{cases}$$
) $\square$

NEXT TIME: DISCUSS BEHAVIOR OF GENERATORS UNDER COBORDISM MAPS.