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Discussing structure of Lee homology (of a knot $K \subseteq S^3$)

LAST TIME: • Showed how to associate

$$\underset{\substack{\text{(orientation} \\ \text{on } \mathcal{D}(K))}}{o} \rightsquigarrow \underset{\substack{\text{(generator} \\ \text{of } \mathcal{C}_{Lee}(\mathcal{D}(K)))}}{s_o}$$

- Showed that for two orientations, $o, \bar{o}$ of $\mathcal{D}(K)$ $[s_o], [s_{\bar{o}}] \neq 0$ in $\mathcal{H}_{Lee}(K)$.

TODAY: • Show $[s_o], [s_{\bar{o}}] \in \mathcal{H}_{Lee}(K)$ are linearly independent
$\Longrightarrow 2 \leq \dim \mathcal{H}_{Lee}(K)$

- Show $\dim \mathcal{H}_{Lee}(K) \leq 2$.
- Understand behavior of $s_o, s_{\bar{o}}$ under 4D cobordisms.

Lemma (3.5 of Rasmussen): Consider the $\mathbb{Z}/4\mathbb{Z}$-q-grading on $\mathcal{C}_{Lee}(\mathcal{D}(K))$ (Recall: Lee differential $\partial^{Lee} = \partial_0 + \partial_4$, so respects $\mathbb{Z}/4\mathbb{Z}$-grading). If $o, \bar{o}$ are an orientation and its reverse, then $q(s_o + s_{\bar{o}}) \equiv q(s_o - s_{\bar{o}}) + 2 \mod 4$.

Proof: Let $\mathcal{C}^e_{Lee} = \text{Span}\{x \in \mathcal{C}_{Lee} \mid q(x) \equiv 1 \mod 4\}$

$\mathcal{C}^o_{Lee} = \text{Span}\{x \in \mathcal{C}_{Lee} \mid q(x) \equiv 3 \mod 4\}$.

and $\iota : \mathcal{C}^e_{Lee} \oplus \mathcal{C}^o_{Lee} \longrightarrow \mathcal{C}^e_{Lee} \oplus \mathcal{C}^o_{Lee}$

be the chain map $\left(\begin{array}{c|c} \text{Id} & 0 \\ \hline 0 & -\text{Id} \end{array}\right)$.

Since $\begin{cases} a = v_- + v_+ \\ b = v_- - v_+ \end{cases}$, we have $\iota(a) = \pm b$ and $\iota(b) = \pm a$.
(Also note $q(v_+) = q(v_-) + 2$).

Indeed ι(any std. gen. in $V^{\otimes k}$)
$= \iota^{\otimes k}($ " $)$,
where $\iota(a) = \pm b$, $\iota(b) = \pm a$.

Since $s_{\bar{o}}$ obtained from s_o by switching labels $a \longleftrightarrow b$, we have

$$\iota(s_o) = \pm s_{\bar{o}}, \text{ implying}$$

$$\overbrace{s_o + \iota(s_o)}^{\text{eliminates odd part}} = s_o \pm s_{\bar{o}} \quad \text{is in } \mathcal{C}^e$$

$$\overbrace{s_o - \iota(s_o)}^{\text{eliminates even part}} = s_o \mp s_{\bar{o}} \quad \text{is in } \mathcal{C}^o$$

$$\Rightarrow q(s_o + s_{\bar{o}}) \equiv q(s_o - s_{\bar{o}}) + 2 \mod 4.$$

Corollary: Since $s_o + s_{\bar{o}}$, $s_o - s_{\bar{o}}$ have different q gradings mod 4,

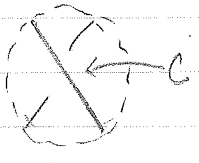
$$[s_o + s_{\bar{o}}] \neq k[s_o - s_{\bar{o}}] \quad \text{for any } k \in \mathbb{Q}$$

$\Rightarrow [s_o]$, $[s_{\bar{o}}]$ are linearly independent. □

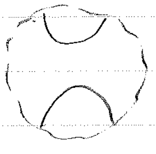
$\underbrace{\text{NOW WE NEED TWO IMPORTANT RESULTS}}$,
- Lee's LES
- Behavior of canonical generators under cobordisms.

THEOREM (3.1 in Lee): Let $\mathcal{D}$ be a link diagram with a distinguished crossing "c", and $\mathcal{D}(*0)$, $\mathcal{D}(*1)$ be the diagrams obtained by replacing c with a "0", "1" resolution:

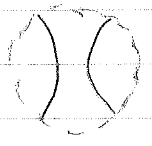
($\mathcal{D}$, $\mathcal{D}(*0)$, $\mathcal{D}(*1)$ agree outside nhd. of c)



$\mathcal{D}$



$\mathcal{D}(*0)$



$\mathcal{D}(*1)$

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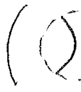
Then there is a LES (long exact sequence)

$$\to \mathcal{H}^{i-1}_{Lee}(\mathcal{D}(*1)) \xrightarrow{\delta} \mathcal{H}^{i}_{Lee}(\mathcal{D}) \to \mathcal{H}^{i}_{Lee}(\mathcal{D}(*0)) \to \mathcal{H}^{i}_{Lee}(\mathcal{D}(*1)) \xrightarrow{\delta} \ldots$$

<u>Proof</u>: Essentially by <u>definition</u> of Lee complex. (and LES for mapping cone).

Recall that $\mathcal{C}_{Lee}\Big($  $\Big) :=$

$\mathrm{MC}\Big(\mathcal{C}_{Lee}\big($  $\big) \xrightarrow[\text{chain map given by saddle cobordism}]{f} \mathcal{C}_{Lee}\big($  $\big)\Big)$

$\uparrow$ mapping cone

<u>DEFN</u>: Let $(\mathcal{C},\partial)$, $(\mathcal{C}',\partial')$ be two chain complexes. $f: (\mathcal{C},\partial) \to (\mathcal{C}',\partial')$ a chain map $(f\partial = \partial' f)$.

Then we can form a new chain complex

$$\begin{tikzcd} \mathrm{MC}(f) := & \mathcal{C} \arrow[rr, bend left, "f"] \arrow[loop below, "\partial"] & \oplus & \mathcal{C}'[-1] \arrow[loop below, "\partial'"] \end{tikzcd}$$

(Here $\mathcal{C}'[-1]_{k+1} = \mathcal{C}'_k$)

$$\partial_{MC(f)} = \left(\begin{array}{c|c} \partial & 0 \\ \hline f & \partial' \end{array}\right)$$

Note that $\mathcal{C}'[-1] \subseteq \mathrm{MC}(f)$ is a subcomplex and $\mathcal{C} \cong \mathrm{MC}(f)/\mathcal{C}'[-1]$ is quotient complex.

$\Rightarrow$ We have a SES (short exact sequence)

$$0 \to \mathcal{C}'[-1] \xrightarrow{\text{inclusion}} \mathrm{MC}(f) \xrightarrow{\text{projection}} \mathcal{C} \to 0$$

The "Snake Lemma" (cf. Hatcher) tells us that we get an associated LES on homology groups. Check: Connecting homomorphism is map induced by chain map, f.

In setting of Lee complex, we obtain:

$$\dots \to \mathcal{H}^{i-1}(\mathcal{D}(*1)) \to \mathcal{H}^{i}(\mathcal{D}) \to \mathcal{H}^{i}(\mathcal{D}(*0)) \to \mathcal{H}^{i}(\mathcal{D}(*1)) \to \dots$$

as desired. □

KEY PROPOSITION (4.1 in Rasmussen) Let $(S^3 \times I, F)$ be a cobordism from $(S^3 \times \{0\}, L_0)$ to $(S^3 \times \{1\}, L_1)$ with no closed components. F induces a well-defined map (up to overall sign)

$$\phi_F : \mathcal{H}_{Lee}(L_0) \longrightarrow \mathcal{H}_{Lee}(L_1)$$

which is filtered of degree $\chi(F)$ $(\phi_F(\mathcal{F}_i) \subseteq \mathcal{F}_{i+\chi(F)}$
Moreover, under this map, if o is an orientation on L_0,

$$\phi_F([\mathfrak{s}_o]) = \sum_{o_I} a_I [\mathfrak{s}_{o_I}],$$

where the sum ranges over all orientations on L_1 compatible (via F) w/ o on L_0, and $a_I \neq 0$ for all o_I.
(Delay proof of this until next time.)

But use (part of) it to prove:

THEOREM (4.2 of Lee): Let $K \subseteq S^3$ be a knot. Then $\dim(\mathcal{H}_{Lee}(K)) \leq 2$ (Combined w/ what we already know, implies $\dim(\mathcal{H}_{Lee}(K)) = 2$.)

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Proof: We will prove that if $L \subseteq S^3$ is a link of either $\ell = 1$ or 2 components, then

$$\dim\left(\mathcal{H}_{Lee}(L)\right) = 2^{\ell}.$$

(By induction on $c = \#$ crossings)
↳ Base case: $c = 0 \Rightarrow$ clear $(\partial^{Lee} = 0)$.

Suppose $\ell = 1$. Choose some crossing of $\mathcal{D}$ and let $\mathcal{D}(*0)$, $\mathcal{D}(*1)$ be two resolutions.

By Lee LES, $rk(\mathcal{H}_{Lee}(\mathcal{D})) \leq rk(\mathcal{H}_{Lee}(\mathcal{D}(*0)) + rk(\mathcal{H}_{Lee}(D(*1)))$.

w/ equality when all maps

$$\mathcal{H}_{Lee}(\mathcal{D}(*0)) \longrightarrow \mathcal{H}_{Lee}(\mathcal{D}(*1))$$

are 0. (Then LES splits into SES's:)

$$0 \longrightarrow \mathcal{H}^i_{Lee}(\mathcal{D}(*1)) \to \mathcal{H}^i_{Lee}(\mathcal{D}) \to \mathcal{H}^i_{Lee}(\mathcal{D}(*0)) \to 0$$

Indeed (check!) if

$$\begin{tikzcd} & \{\mathcal{A}^i\} \arrow[dr] & \\ \{\mathcal{C}^i\} \arrow[ur, "\delta^i"] & & \{\mathcal{B}^i\} \arrow[ll] \end{tikzcd}$$

connecting homomorph.

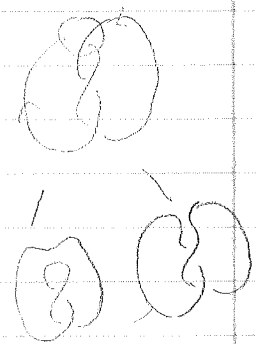
is a LES (over a field)

and $rk(\delta) := \sum_i rk(\delta^i)$

then

$$\dim(\underset{\underset{\oplus \mathcal{B}^i}{\uparrow}}{\mathcal{B}}) = \dim(\underset{\underset{\oplus \mathcal{A}^i}{\uparrow}}{\mathcal{A}}) + \dim(\underset{\underset{\oplus \mathcal{C}^i}{\uparrow}}{\mathcal{C}}) - 2 rk(\delta).$$

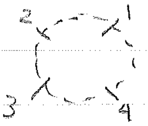
(Sketch: We have SES's



$$0 \to \operatorname{coker}(\delta^{i-1}) \to \mathcal{B}^i \to \ker(\delta^i) \to 0$$

Now use
- $\dim(\ker(\delta^i)) + \operatorname{rk}(\delta^i) = \dim(\mathcal{C}^i)$
- $\operatorname{rk}(\delta^i) + \dim(\operatorname{coker}(\delta^i)) = \dim(\mathcal{A}^i)$)

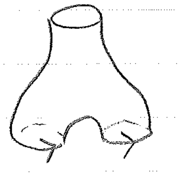
① Now note that if $\mathcal{D}$ is a diagram for a knot, then one of $\mathcal{D}(*0)$, $\mathcal{D}(*1)$ is a diagram for a knot and the other for a 2-component link (Check!



Indeed, can also see <u>oriented</u> resolution is 2-component link.)

② Similarly, if $\mathcal{D}$ is a nonsplit diagram for a 2-comp. link, $\mathcal{D}(*0)$ and $\mathcal{D}(*1)$ are both knots. (Check! Similar to above.). (If $\mathcal{D}$ split → tensor product).

In either case, $\mathcal{D}(*0)$, $\mathcal{D}(*1)$ have one fewer crossing.



For saddle cobordism: orientation only extends from "pant leg" end if oriented as  or reverse, ∃ a unique extension (if so) since connected.

By Rasmussen's key Propn. 4.1, the map

$$\mathcal{H}_{Lee}(\mathcal{D}(*0)) \longrightarrow \mathcal{H}_{Lee}(\mathcal{D}(*1))$$

has rank at least 2 (image of $\mathfrak{s}_o, \mathfrak{s}_{\bar{o}}$ if $\mathcal{D}(*0)$ a knot, image of two of "relative" orientations if $\mathcal{D}(*1)$ a knot) in case ①

So: Case ① $\longrightarrow$ $\dim(\mathcal{H}_{Lee}(\mathcal{D})) \leq 2+4-4 = 2$

Case ② $\longrightarrow$ $\dim(\mathcal{H}_{Lee}(\mathcal{D})) \leq 2+2 = 4$.

(Gap: Haven't shown that $4 \leq \dim(\mathcal{H}_{Lee}(\overset{\text{2 component}}{L^{\circlearrowleft}}))$. Follows from gr.)