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Last time stated

Key Proposition (4.1 from Rasmussen): A cobordism

$$(S^3, L_0) \xrightarrow{(S^3\times I,\, F)} (S^3, L_1)$$

with no closed components induces a map (well-defined up to overall sign)

$$\phi_F : \mathcal{H}_{Lee}(L_0) \longrightarrow \mathcal{H}_{Lee}(L_1),$$

which is filtered of degree $\chi(F)$. Also:

$$\phi_F([s_0]) = \sum_{o_I} a_I [s_{o_I}]$$

(o_I a compatible orientation, $a_I \neq 0$).

Proof: $\boxed{\text{Defn. of maps}}$ Decompose F into a sequence of elementary cobordisms, represented by a "Carter-Saito movie" (reference: original Carter-Saito paper, more comprehensive book)

"Critical frames" in movie:

1. Reidemeister moves: RI, II, III $\Rightarrow \rho_{I,II,III}$
2. Morse moves: Birth (0-handle) $\Rightarrow \iota$
   Death (2-handle) $\Rightarrow \varepsilon$
   Saddle (1-handle) $\Rightarrow \Delta, m$

Maps associated to these moves are precisely the ones Lee defines.

$\boxed{\text{Well-defined up to sign}}$ "Carter-Saito movie moves"

(Bar-Natan, M. Jacobsson).

Tells us that a composition of such maps has filtered degree $\chi(F)$.

Reid. moves $\to$ Degree 0 (in fact, filt. homot. equivs.)

Birth/Death $\to$ Degree 1

$\iota :$ $\longrightarrow$ ○ $\qquad \iota(1) = v_+ \qquad q(v_+) = 1.$

$\varepsilon :$ ○ $\longrightarrow$ $\qquad \varepsilon(v_-) = 1 \qquad q(v_-) = -1.$

Saddle $\to$ Degree -1 (exactly the reason Khovanov puts q-degree shifts in his complex).

$m :$ $v_{++} \to v_+$; $v_{+-}, v_{-+} \to v_-$; v_{--} $\qquad q(v_{++}) = 2$, $q(v_{+-}) = q(v_{-+}) = 0$ $\qquad q(v_+) = 1$, $q(v_-) =$

$\Delta :$ $v_+ \to v_{++}$, $v_+ \to v_{+-}$, $v_+ \to v_{-+}$; $v_- \to v_{--}$

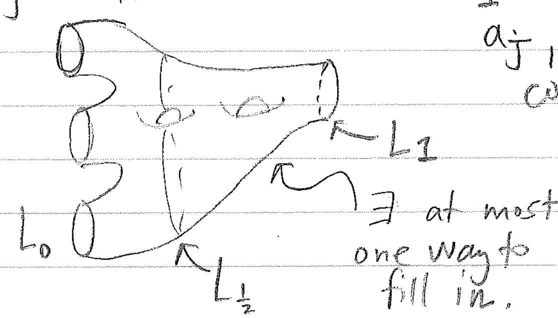
$\boxed{\text{Map } \phi_F \text{ is a sum (w/ nonzero coeffs.) over compatible orientations:}}$

Elementary cobordisms $\Longrightarrow$ check.

By induction on # of elem. cobordisms

(Cut cobordism along an intermediate nontrivial slice.

Then for each compatible orientation o_I on L_1, corresponding coefficient is $a_I = a_J \cdot a_K$ where a_J, a_K are nonzero coefficients associated to orientation on $L_{\frac{1}{2}}$)



(p. 2)

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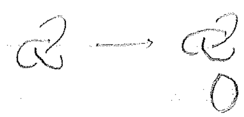
Example of calc. for elementary cobordism:

F is a 0-handle addition:

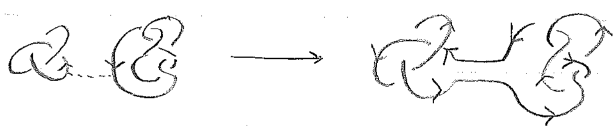
$$\phi_F(\mathfrak{s}_o) = z(\mathfrak{s}_o) = \mathfrak{s}_o \otimes v_+$$
$$= \mathfrak{s}_o \otimes \tfrac{1}{2}(a-b)$$
$$= \tfrac{1}{2}(\underbrace{\mathfrak{s}_o \otimes a}) - \tfrac{1}{2}(\underbrace{\mathfrak{s}_o \otimes b})$$



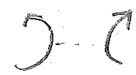
nonzero coefficients. one orientation The other orientation.

F is a merge saddle, locally oriented  :

Note: $\exists$ a unique orientation on cobordism



Using Lemma about coherent orientations, strands have same label. But $m(a\otimes a) = 2a$
$m(b \otimes b) = -2b$.

F is a merge saddle, locally oriented  :
Note: $\nexists$ a compatible orientation on F

$\square$

One important property of s(K):

<u>Proposition</u> (3.10 in Rasmussen): If $\overline{K}$ is the mirror of K (change all crossings in a diagram),

K in 3-mfd. w/ opposite orientation

$$s(\overline{K}) = -s(K)$$

Proof: (Exercise) The filtered chain cx. $\mathcal{C}_{Lee}(\bar{K})$ is isomorphic to $(\mathcal{C}_{Lee}(K))^*$ (+ some algebra).

$q(x^*) = -q(x)$.

(See original Khovanov paper: $\Delta^* = \bar{m}$ and $m^* = \bar{\Delta}$). $\square$

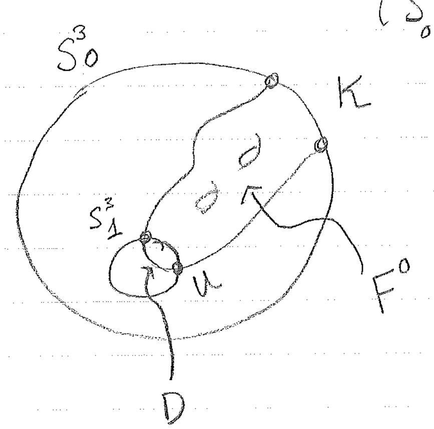
Theorem (1 of Rasmussen): $|s(K)| \leq 2g_4(K)$.

Proof: Let $F \hookrightarrow B^4$ be a properly-imbedded surface w/ $\partial F = K$ of genus $= g_4(K)$.

By removing a small (neighborhood of a) disk in the interior of F, one obtains a cobordism

$$(S^3_0, K) \xrightarrow{(S^3 \times I,\, F^0)} (S^3_1, U)$$

$F^0 = F \setminus D$



First filtration level $\mathcal{F}_K$ where induced inclusion map on homology is $\neq 0$.

Let $x \in \mathcal{H}_{Lee}(K) \setminus \{0\}$ have $s(x) = s_{max}(K)$.

By Key Prop., $\phi_F : \mathcal{H}_{Lee}(K) \to \mathcal{H}_{Lee}(U)$ is filt. of degree $\chi(F^0) = -2g(F) = -2g_4(K)$

$$s(\phi_F(x)) \geq s_{max}(K) - 2g_4(K).$$

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But $\phi_F(x) \neq 0$ (also by Key Prop.)
and $s_{max}(\mathcal{U}) = 1$ by direct calculation.
So

$$s_{max}(K) - 2g_4(K) \leq s(\phi_F(x)) \leq 1.$$

$$\Longrightarrow \quad \underbrace{s(K) - 2g_4(K) = (s_{max}(K) - 1) - 2g_4(K)}_{\leq 0}$$

$$\Longrightarrow \quad s(K) \leq 2g_4(K).$$

To see $-s(K) = s(\bar{K}) \leq g_4(\bar{K}) = g_4(K)$,
apply same argument as above to
$\bar{F} \subseteq \bar{S^3 \times I}$
reverse orientation. Obtain:

$$\phi_{\bar{F}} : \mathcal{H}_{Lee}(\bar{K}) \to \mathcal{H}_{Lee}(\mathcal{U})$$

(filtered of degree $-2g_4(\bar{K}) = -2g_4(K)$).
So $-s(K) \leq 2g_4(K)$. $\square$.

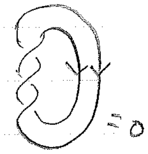
Last step to Milnor conjecture:

Computation of $s(T_{p,q})$: Note $T_{p,q}$ is positive knot (braid).

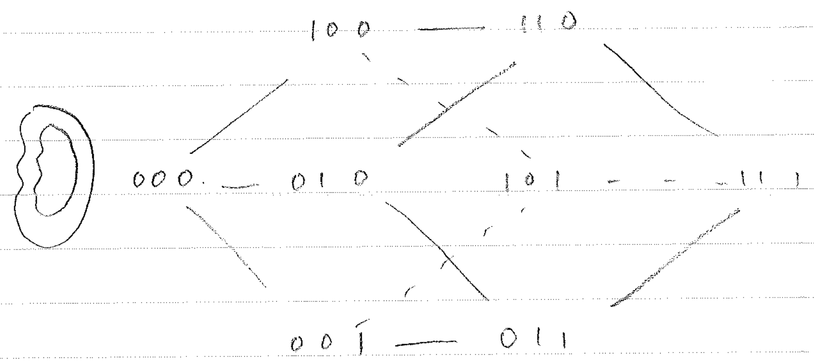
Theorem (4 in Rasmussen): If K is a positive knot (all crossings: $\nearrow\!\!\!\!\nwarrow$), then

$$s(K) = 2g_4(K) = 2g(K).$$

Proof: Both oriented resolutions are in the lone vertex @ far left end of cube.

Example: 

Oriented resolution (for either o or $\bar{o}$) = "000" resolution.



** ⇒ Since there are no edges coming into this vertex, there are no generators homologous to s_o, $s_{\bar{o}}$.

Compute: $s_{min}(K) = s(s_o) = s(s_{\bar{o}})$
$= q(\underbrace{v_- \otimes \ldots \otimes v_-}_{\vec{v}})$

$$= \underbrace{p(v_- \otimes \ldots \otimes v_-)}_{-(\#\text{ components of resolution})} + \underbrace{|\vec{v}|}_{0} + \underbrace{n_+}_{=\#\text{ positive crossings}} - 2\underbrace{n_-}_{0}$$

S = Seifert surface from Seifert algorithm.

$-(\#$ components of resolution$)$ $= -(\#$ 0-handles of $S)$

$\#$ positive crossings $= \#$ crossings $= (\#$ 1-handles of $S)$

$= -\chi(S)$

So $s(K) = 1 - \chi(S)$
$= 2g(S)$

We have $s(K) = 2g(S) \leq 2g_4(K) \leq 2g(K) \leq 2g(S)$ □