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Recall: Last time we discussed how to associate

$K \subseteq Y^3 \rightsquigarrow$ marked $\Rightarrow$ doubly-pointed Heegaard diagram for $K \subseteq Y^3$.

knot (K); closed, connected, oriented 3-mfd (Y^3).

Today: Discuss how to construct a $\mathbb{Z}$-filtered complex from this data. $\boxed{\text{RESTRICT TO: } K \subseteq Y^3 \text{ NULLHOMOLOGOUS}}$ (can be relaxed, but simplest for now).

START: Data of $(\Sigma, \vec{\alpha}, \vec{\beta}, w, z)$ compatible w/ K. (genus g)
+ additional data of generic (family of) almost-complex structure(s) on Σ and compatible symplectic structure ω.

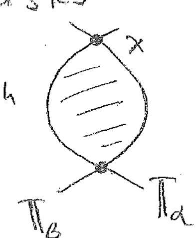
(Forget K momentarily, i.e. FORGET z BASEPOINT).

Look @ Lagrangian Floer homology of $\mathbb{T}_\alpha, \mathbb{T}_\beta \subset \mathrm{Sym}^g(\Sigma)$ ($\mathbb{Z}/2\mathbb{Z}$ coefficients)

Generators of chain complex, $\widehat{CF}(Y)$:
Elements of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

Differentials:
Let $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

Recall: $\pi_2(x,y)$ = homotopy classes of Whitney disks $D^2 \hookrightarrow \mathrm{Sym}^g(\Sigma)$



$\boxed{\text{When } g > 1}$:

$\pi_2(x,y)$ is either $\begin{cases} \text{empty (if } \varepsilon(x,y) \neq 0) \\ \cong \mathbb{Z} \oplus H^1(Y) \text{ (if } \varepsilon(x,y) = 0). \end{cases}$

Will discuss at great length soon.

(The idea: If $\varepsilon(x,y) = 0$, then $\exists$ some $\phi_0 \in \pi_2(x,y)$.
All other $\phi \in \pi_2(x,y)$ can be obtained from ϕ_0 by ① connect-sum S^2 gen. of $\pi_2(\mathrm{Sym}^g(\Sigma))$ and ② appending periodic domains.)

generates $\mathbb{Z}$-summand

generates $H^1(Y)$-summand.

$$\hat{\partial} x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \; \sum_{\substack{\{\phi \in \pi_2(x,y) \,|\, \\ \mu(\phi) = 1, \\ n_w(\phi) = 0\}}} \widehat{\mathcal{M}}(\phi) \cdot y.$$

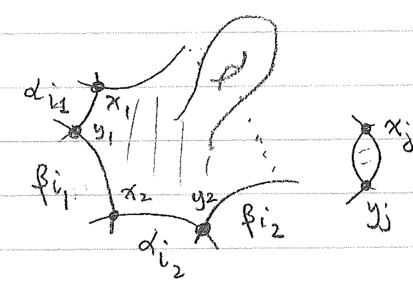
where $\widehat{\mathcal{M}}(\phi)$ is moduli space of holomorphic representatives of ϕ, quotiented by $\mathbb{R}$-action.

<u>Remarks</u>: ① Picking a generic (family of) almost complex structure(s) guarantees that the dimension of $\mathcal{M}(\phi)$ agrees with $\mu(\phi)$ (Maslov index = "expected dimension"
= dimension of <u>transverse</u> intersection in moduli space.

How you should think of this: Recall that associated to every $\phi \in \pi_2(x,y)$ is a corresponding <u>domain</u> in Σ

$$\begin{array}{ccc} F & \overset{\text{immersed}}{\hookrightarrow} & \Sigma \\ {\scriptstyle g:1} \downarrow & & \\ D^2 & \hookrightarrow & \mathrm{Sym}^g(\Sigma) \end{array}$$

A <u>domain</u> representing $\phi \in \pi_2(x,y)$ is an immersion of a surface F into Σ that is topologically a g-fold branched cover of D^2.



Such a domain admits a holomorphic rep. if $\exists$ a complex structure on F pulled back from a map $F \hookrightarrow \Sigma$ that is simultaneously a complex structure pulled back from branched cover of D^2!

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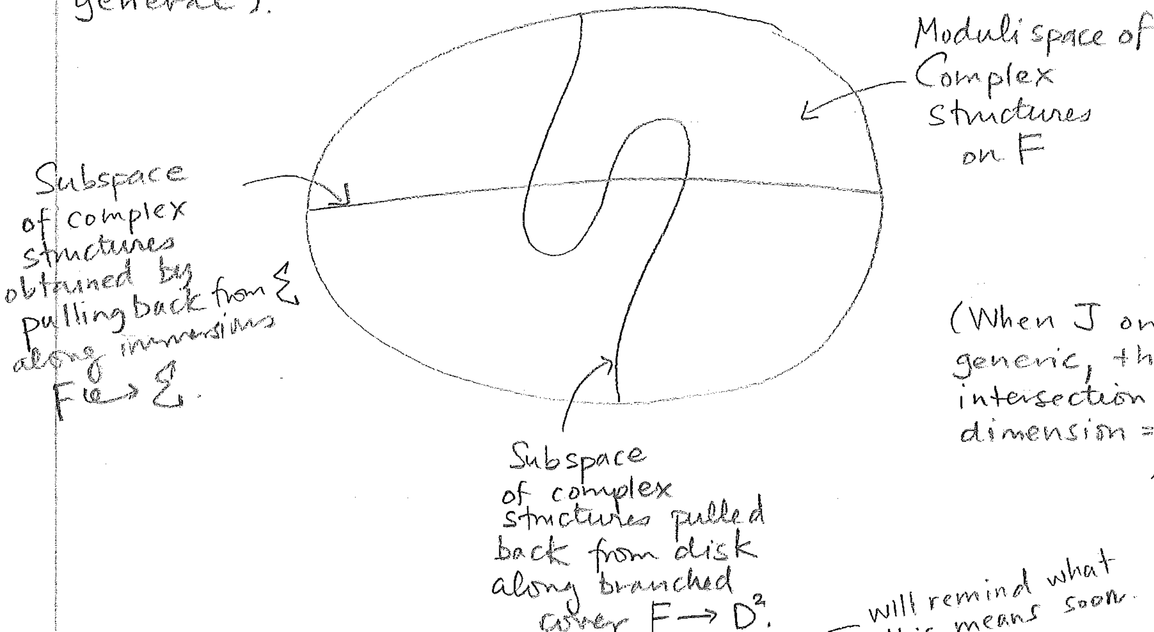
So, think of this as an intersection theory problem in moduli space (Beware: picture is misleading, since neither of spaces is a submanifold in general).



(When J on Σ generic, this intersection has dimension $= \mu(\phi)$).

will remind what this means soon.

② Making Σ <u>weakly admissible</u> ensures that for each pair $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, only finitely many $\phi \in \pi_2(x,y)$ with $\mu(\phi) = 1$ have all non-negative coefficients (hence can support holomorphic representative).

When $H^1(Y) = 0$, don't need to worry about this.

① + ② ensures that $\widehat{\mathcal{M}}(\phi)$ is finite $\Rightarrow$ $\hat{\partial}$ makes sense. (w/ some extra work, relying on Gromov's compactness theorem, on ways holomorphic disks can degenerate, Ozsváth–Szabó show that $\hat{\partial}^2 = 0$).

NOW REMEMBER $K \subseteq Y$ (i.e., remember z basepoint)

nullhomologous

<u>Claim</u>: $\exists$ a natural assignment

$$A_{w,z} : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathbb{Z}$$

called the <u>Alexander grading</u> (when $K \subseteq S^3$, this agrees with exponent of T in symmetrized Alexander polynomial) with the property that if $\phi \in \pi_2(x,y)$ then

$$A_{w,z}(x) - A_{w,z}(y) = n_z(\phi) - n_w(\phi).$$

Note: Intersection positivity of complex submanifolds tells us that if ϕ is a domain contributing to the differential $\hat{\partial}$ (hence ϕ admits a holomorphic representative), then $n_z(\phi) \geq 0$ (Recall: we restrict to $n_w(\phi) = 0$)

$\Rightarrow \widehat{CF}(Y)$ is a bounded <u>filtered</u> complex

$$0 = \mathcal{F}_n \subseteq \mathcal{F}_{n+1} \subseteq \dots \subseteq \mathcal{F}_{N-1} \subseteq \mathcal{F}_N = \widehat{CF}(Y)$$

where $\mathcal{F}_m := \mathrm{Span}\{x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \mid A_{w,z}(x) \leq m\}$

Since $A_{w,z}(y) \leq A_{w,z}(x)$ whenever

$$\langle \hat{\partial}(x), y \rangle \neq 0,$$

each $\mathcal{F}_m$ is a subcomplex.

The <u>knot Floer homology</u> $\widehat{HFK}(Y,K)$ is the homology of the <u>associated graded</u> complex of this filtered complex (i.e., the E^1 page)

Concretely: $\widehat{CFK}(Y,K)$ has

Generators $\longleftrightarrow$ elements of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$

Differential:

$$\hat{\partial}_K(x) = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\phi \in \pi_2(x,y) \mid \mu(\phi)=1,\ n_w(\phi) = n_z(\phi) = 0\}} \widehat{\mathcal{M}}(\phi) \cdot y$$

Properties of μ:

$\mu(\phi * \psi) = \mu(\phi) + \mu(\psi)$

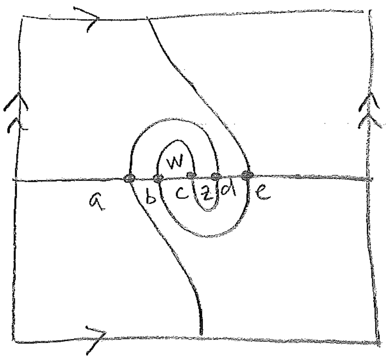
$\mu(-\phi) = -\mu(\phi)$

$\mu(\text{bigon}) = 1$, $\mu(\Sigma_1) = 2$.

(p. 3)

d
e ∘∘c∘a
b∘

Example:



Doubly-pointed H.D.
Figure-8 knot

2-bridge knot
$K_{5,2}$.

BEWARE: I MAY HAVE DESCRIBED $K_{p,-q}$.

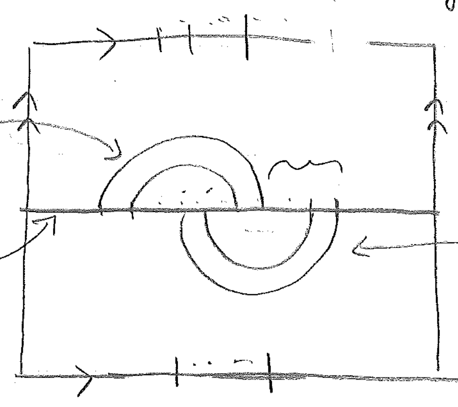
FACT 1: 2-bridge knots are classified by the oriented homeomorph. type of their double-branched covers (lens spaces).

So $\Sigma(K_{5,2}) = L(5,2)$.

Because of known homeos., can arrange $1 \leq q < \frac{p}{2}$.

Can be seen from Schubert normal form → FACT 2: A doubly-pointed Heegaard diagram for $K_{p,q}$ can be obtained by

q caps (uses 2q points)



q cups (uses 2q points)

α curve

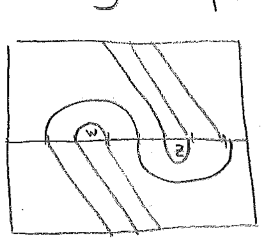
Mark p points on here.

p-2q points here

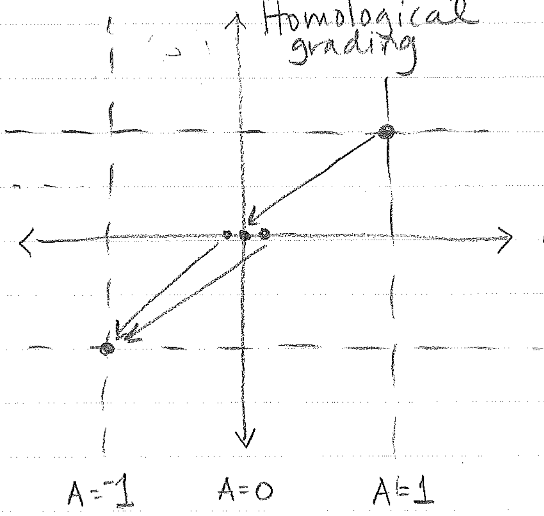
Connect remaining p-2q points along α curve in order along top/bottom. Then "lock in place" w/ w/z basepoints in innermost cap/cup.

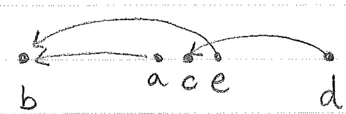
Example:

$K_{7,2}$



Calculation of $\widehat{CF}(S^3, K_{5,2})$:





$\mathrm{rk}\left(\widehat{CFK}(S^3, K_{5,2})\right) = 5$

Two gradings: Homological (Maslov) & Filtration (Alexander) grading.

$\hat{\partial}_K = 0 \quad \forall \quad x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ since $M(x) - M(y) = A(x) - A(y)$ for all pairs $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

$M(x) - M(y) = \mu(\phi)$ for $\phi \in \pi_2(x,y)$ w/ $n_w(\phi) = 0$.

(in particular $\nexists$ $\phi \in \pi_2(x,y)$ w/ $\mu(\phi) = 1$, $n_w(\phi) = 0$ and $n_z(\phi) = 0$)