3/19/14

Recall: K nullhomologous $\subseteq Y^3$ c.c.o. 3-mfd. $\rightsquigarrow$ Chain complex $\widehat{CF}(Y^3)$, filtered by Alexander grading $A_{w,z} : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathbb{Z}$.

Asserted that $A_{w,z}(x) - A_{w,z}(y) = n_z(\phi) - n_w(\phi)$ for any $\phi \in \pi_2(x,y)$, but didn't define $A_{w,z}$.

Recall: $\widehat{CF}(Y^3) = \bigoplus_{s \in Spin^c(Y)} \widehat{CF}(Y,s)$

~~Heegaard-Floer~~ Chain complex splits according to $Spin^c$ structures on Y.

Great reference for geom. view of $Spin^c(Y)$: Turaev's paper w/ "Torsion" in title $\to$ $Spin^c$ structure = "Homology class" of non-vanishing vector fields on Y.

Important fact for us: There is an affine identification

$$Spin^c(Y) \longleftrightarrow H_1(Y) \cong H^2(Y).$$

(Just means: $\forall\ s, s' \in Spin^c(Y)$, $s - s' \in H_1(Y)$).

realized concretely as follows:

① The basepoint, w, determines a map $s_w : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow Spin^c(Y)$

(explicitly construct a vector field by altering gradient vector field in a neighborhood of flow lines corresponding to w and intersection pt. in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$).

Affine identification w/ $H_1(Y)$:

Given $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, define

Section 2.6, O-S "Hol. disks ... closed 3-manifolds"

$s_w(y) - s_w(x) = \varepsilon(x,y) \in H_1(Y)$ by:

- Connect $x \to y$ along α curves (γ_a). $x \to y$ along β curves (γ_b).
- Resulting 1-cycle $\gamma_a - \gamma_b$ represents an element of $H_1(\Sigma) \hookrightarrow H_1(Y)$

Check: Independent of choice of paths $x \to y$, since

$$H_1(Y) \cong H_1(\Sigma) \Big/ \mathrm{Span}\{[\alpha_1], \dots, [\alpha_g], [\beta_1], \dots, [\beta_g]\}$$

Think of $\varepsilon(x,y)$ as algebro-topological (first, obvious) obstruction to the existence of a Whitney disk connecting $x \to y$.

$\varepsilon(x,y) = 0 \in H_1(Y)$ (i.e., $s_w(y) = s_w(x)$)

iff

$[\gamma_a - \gamma_b] = 0 \in H_1(Y)$

iff

$\exists\, a_i \in \mathbb{Z},\ b_i \in \mathbb{Z}$ s.t.

$$\underbrace{\gamma_a - \gamma_b + \sum a_i \alpha_i + \sum b_i \beta_i}_{\gamma_a' - \gamma_b' \text{ bounds a domain representing } \phi \in \pi_2(x,y).} = 0 \text{ in } H_1(\Sigma) \cong \pi_1(\mathrm{Sym}^g(\Sigma))$$

NOW REMEMBER DATA $K \subseteq Y^3$ (i.e., basepoint z)

Define $\underline{\mathrm{Spin}^c(Y,K)} := \mathrm{Spin}^c(Y_0(K))$ ← 0-surgery on K.