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$K \subseteq Y^3$ nullhomologous (+ a choice of Seifert surface F) $\rightsquigarrow$ doubly-ptd. Heegaard diag. $(\Sigma, \alpha, \beta, w, z)$ $\rightsquigarrow$ Heegaard-Floer chain comp. $\widehat{CF}(Y)$, filtered by $A_{w,z} : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathbb{Z}$

(Last time: $\mathrm{Spin}^c(Y)$)

Today: Understand Alexander grading better

** For simplicity, assume also that $b_1(Y) = 0$.

Then $A_{w,z} : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \mathbb{Z}$ is defined as follows: We have $s_w : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathrm{Spin}^c(Y)$

Consider

$$c_1 : \mathrm{Spin}^c(Y_0(K)) \longrightarrow H^2(Y_0(K))$$
$$\mathfrak{s} \longmapsto \mathrm{P.D.}(\underbrace{\mathfrak{s} - \bar{\mathfrak{s}}}_{\in H_1(Y_0(K))})$$

Then

$$A_{w,z} := \frac{1}{2} \langle c_1(\mathfrak{s}^\circ_w(x)), [\hat{F}] \rangle$$

capped off Seifert surface for K.

where $\mathfrak{s}^\circ_w \in \mathrm{Spin}^c(Y_0(K))$ is a canonical choice of Spin^c structure on $Y_0(K)$ <u>extending $\mathfrak{s}_w(x)$</u> $\in \mathrm{Spin}^c(Y)$

defined concretely as follows.

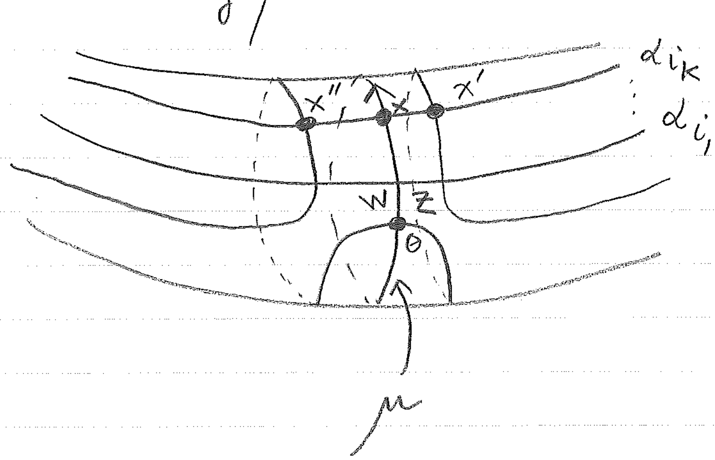
Let $(\Sigma, \vec{\alpha}, \vec{\beta} = \vec{\beta}_0 \cup \mu, w, z)$ be a doubly-pointed Heegaard diagram for (Y, K)

Replace μ w/ λ = longitude for K

("0"-framed, i.e. bounds $F \subseteq Y - N(K)$)
to get a Heegaard diagram

$$(\Sigma', \vec{\alpha}, \vec{\gamma} = \vec{\beta}_0 \cup \lambda) \quad \text{for } Y_0(K)$$

"Wind λ along μ":



to ensure that Σ' is weakly admissible, as long as Σ was.

(Recall: Σ is weakly-admissible for all spin^c structures if every periodic domain has both pos./neg. coefficients)

domain bounded by $\vec{\alpha}, \vec{\beta}$ curves missing w

This is $s_w : \mathbb{T}_\alpha \cap \mathbb{T}_\gamma \to \mathrm{Spin}^c(Y_0(K))$

Then $s^0_w(\vec{x}) := s_w(\vec{x}') = s_w(\vec{x}'')$
$\in \mathrm{Spin}^c(Y_0(K))$

Note that x' connected to x (and lone intersection pt θ between λ and μ) by an obvious triangle.

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Oz-Sz have "c_1-evaluation" formula (Section 6.1, O-S "4-mfd. invariants") that allows us to compute

$$A_{w,z}(x) := \frac{1}{2}\langle c_1(\mathfrak{s}^0_w(x)), [\hat{F}]\rangle$$

by first finding a periodic domain representing $[\hat{F}]$ on $(\Sigma', \vec{\alpha}, \vec{\gamma}, w)$.

(Good exercise)

In the case $\mathfrak{s}_w(x)$ is torsion (more precisely, $c_1(\mathfrak{s}_w(x)) := \mathfrak{s}_w(x) - \overline{\mathfrak{s}_w(x)} \in H_1(Y)$ is torsion)

↳ Note: If we are restricting to case $b_1(Y)=0$, all Spin^c structures are tors.

we understand Alexander grading in a (perhaps) more concrete way as follows:

$\exists$ a unique $t_0 \in \mathrm{Spin}^c(Y_0(K))$ extending $\mathfrak{s} = \mathfrak{s}_w(x)$ with the property that

$$\langle c_1(t_0), [\hat{F}]\rangle = 0 \quad \text{for any } F \text{ w/ } \partial F = K.$$

(The point: $H_1(Y - N(K)) \xrightarrow{\text{incl.}} H_1(Y)$

$$\begin{array}{c} H_1(Y-N(K)) \\ \text{incl.}\ \ \cong \\ H_1(Y_0(K)) \cong H_1(Y) \times \mathbb{Z} \end{array}$$

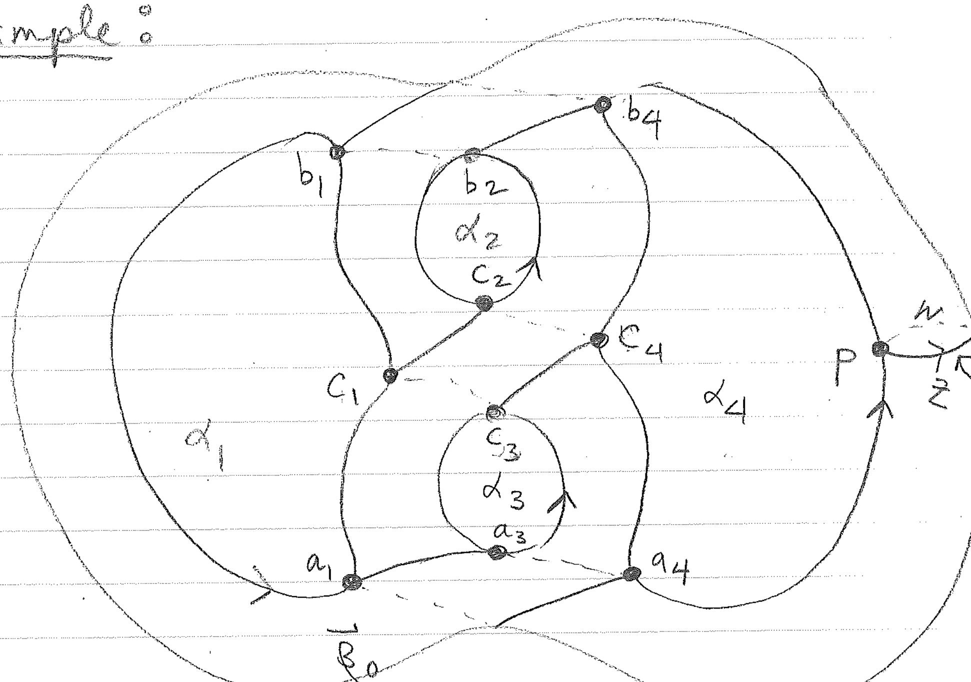
and μ is a canonical generator of $\mathbb{Z}$.

A marked Heegaard diagram provides a

Remark: $\mathfrak{s}$ torsion ensures that in the decomp. $c_1(\mathfrak{t}_0) = \alpha + m \cdot PD[\mu]$, $\langle \alpha, [\hat{F}] \rangle = 0$, so $\langle c_1(\mathfrak{t}_0), [\hat{F}] \rangle$ is independent of choice of F w/ $\partial F = K$.

"canonical" way to restrict $\mathfrak{s}$ to $Y - N(K)$ then extend to $Y_0(K)$.)

Example:



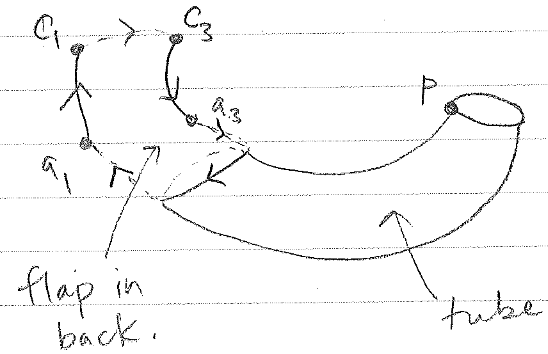
$$\begin{array}{cc} & \overbrace{\beta_1 \quad \beta_2 \quad \beta_3}^{\vec{\beta}_0} \quad \beta_4 \\ \vec{\alpha}_0 \left\{ \begin{array}{c} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{array} \right. & \begin{pmatrix} b_1 & c_1 & a_1 & 0 \\ b_2 & c_2 & 0 & 0 \\ 0 & c_3 & a_3 & 0 \\ b_4 & c_4 & a_4 & p \end{pmatrix} \\ \alpha_4 & \quad \vec{x}_0 \qquad\qquad \vec{y}_0 \end{array}$$

Any elt. of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ uniquely specified by an elt. of $\mathbb{T}_{\alpha_0} \cap \mathbb{T}_{\beta_0}$.

Consider "$\varepsilon(\underbrace{(b_2, c_3, a_1)}_{\vec{x}_0}, \underbrace{(b_2, c_1, a_3)}_{\vec{y}_0})$" (class in $H_1(Y - N(K)) \to$ represents some multiple of μ

$\varepsilon(\vec{x}_0, \vec{y}_0)$ is homologous on Σ to $m \cdot \mu$ for some $m \in \mathbb{Z}$ (bounds a domain ϕ)
$n_z(\phi) - n_w(\phi) = m$

Ex.:
$\varepsilon(\vec{x}_0, \vec{y}_0) = \mu$



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What about <u>Absolute</u> $A_{w,z}$ grading? <u>SYMMETRY</u>

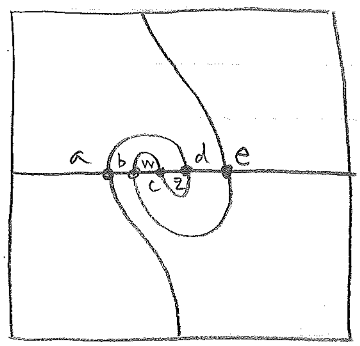
Recall that if $\mathfrak{s} \in Spin^c(Y)$ has $c_1(\mathfrak{s})$ torsion any $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ w/ $\mathfrak{s}_w(x) = \mathfrak{s}$ has a well-defined $\mathbb{Q}$-grading (coming from viewing $Y = \partial W$), that is absolute $\mathbb{Q}$-lift of relative Maslov (homological) grading.

<u>Claim:</u> For Σ, a 2-pointed Heegaard diagram, $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with torsion $\mathfrak{s}_w(x) \in Spin^c(Y)$,

$$A_{w,z}(x) = \frac{1}{2}\left(M_w(x) - M_z(x)\right)$$

absolute $\mathbb{Q}$-grading with respect to w,z basepoints.

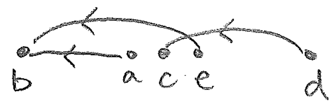
<u>Example:</u>



$M_w(e) = M_w(c) = M_w(a) = 0$

$M_w(d) = 1, \quad M_w(b) = -1$

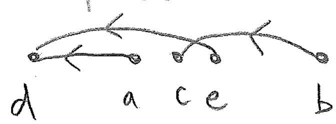
with w basepoint:



$M_z(e) = M_z(c) = M_z(a) = 0$

$M_z(b) = 1, \quad M_z(d) = -1.$

with z basepoint:



$\Rightarrow A_{w,z}(e, c, a) = 0$
$A_{w,z}(d) = 1, \quad A_{w,z}(b) = -1.$

Now we can define $\tau(\Sigma, \alpha, \beta, w, z) \in \mathbb{Z}$ for (Σ, α, β) representing $Y = S^3$ (or any $\mathbb{Z}HS^3$)
(in general, would have $\tau : \text{Torsion } Spin^c(Y) \to \mathbb{Z}$)

Will briefly discuss next time why τ is an invariant of $(\underline{Y, K})$ (doesn't depend on 2-ptd. H.D.)

Let $K \subseteq S^3$, and $(\Sigma, \alpha, \beta, w, z)$ a 2-ptd. H.D.

We have :

$$0 = \mathcal{F}_n \subseteq \mathcal{F}_{n+1} \subseteq \ldots \subseteq \mathcal{F}_N = \widehat{CF}(S^3)$$

where $\mathcal{F}_m := \mathrm{Span}_{\mathbb{F}} \{ x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \mid A_{w,z}(x) \leq m \}$

We also have (directed system of) inclusion maps

$$i_m : \mathcal{F}_m \longrightarrow \widehat{CF}(\Sigma)$$

So we can ask about rank of $(i_m)_* : H_*(\mathcal{F}_m) \to \widehat{HF}(S^3) \cong \mathbb{F}$

<u>Remark</u>: (1) For m sufficiently $\left\{ \begin{matrix} \text{small} \\ \text{large} \end{matrix} \right\}$ $(i_m)_*$ is $\left\{ \begin{matrix} \text{zero} \\ \text{an isomorphism} \end{matrix} \right\}$.

(2) Since we have commutative diagrams

$$\begin{tikzcd} \mathcal{F}_m \arrow[r, "i_{m,M}"] \arrow[dr, "i_m"'] & \mathcal{F}_M \arrow[d, "i_M"] \arrow[dl, phantom, "\circlearrowright"] \\ & \widehat{CF}(S^3) \end{tikzcd} \qquad \forall\ m, M \in \mathbb{Z}$$

we know that if $(i_M)_* = 0$ for some $M \in \mathbb{Z}$ then $(i_m)_* = 0$ $\forall$ $m \leq M$

and

if $(i_m)_*$ is surjective for some $m \in \mathbb{Z}$, then $(i_M)_*$ is surjective $\forall$ $M \geq m$.

$\Longrightarrow$ For any bounded $\mathbb{Z}$-filtered complex as above, there are two critical integers.

$$0 = \mathcal{F}_n \subseteq \mathcal{F}_{n+1} \subseteq \ldots \subseteq \mathcal{F}_N = \mathcal{C}$$

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$$\tau_{min}(\mathcal{F}) := \min_{m \in \mathbb{Z}} \left\{ (i_m)_* : H_*(\mathcal{F}_m) \to H_*(\mathcal{C}) \right\}$$
is nonzero.

$$\tau_{max}(\mathcal{F}) := \min_{m \in \mathbb{Z}} \left\{ (i_m)_* : H_*(\mathcal{F}_m) \to H_*(\mathcal{C}) \right\}$$

is surjective.

Note that since $\widehat{HF}(S^3) \cong \mathbb{F}$, in the case of interest to us (i.e., $H_*(\mathcal{C})$ is rank <u>one</u>) $\tau_{min}(\mathcal{F}) = \tau_{max}(\mathcal{F})$.

<u>Definition</u>: Let $(\Sigma, \vec{\alpha}, \vec{\beta}, w, z)$ be a 2-ptd. Heegaard diagram for $K \subseteq S^3$, $\mathcal{F}$ the corresponding filtered chain complex.

$$\tau(\Sigma, \alpha, \beta, w, z) := \tau_{min}(\mathcal{F}) = \tau_{max}(\mathcal{F}).$$

(For general nullhomologous $K \subseteq Y^3$, for each torsion $\mathfrak{s} \in Spin^c(Y)$ $\exists$ a distinguished element $\xi_{\mathfrak{s}} \in \widehat{HF}(Y, \mathfrak{s})$, and we can ask for first $m \in \mathbb{Z}$ such that $\xi_{\mathfrak{s}} \in im(i_m)_*$.

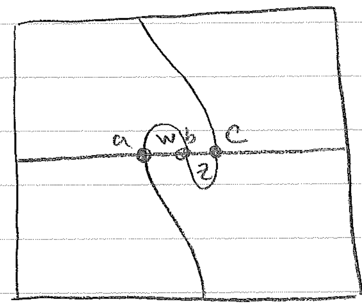
$$(i_m)_* : H_*(\mathcal{F}_m) \longrightarrow \widehat{HF}(Y, \mathfrak{s})$$

So we get a τ invariant for each torsion $Spin^c$ structure.)

Next time: We'll argue quickly that τ is an invariant of $K \subseteq Y$ (by explaining why filtered chain homotopy type of $\mathcal{F}$ is invt.)

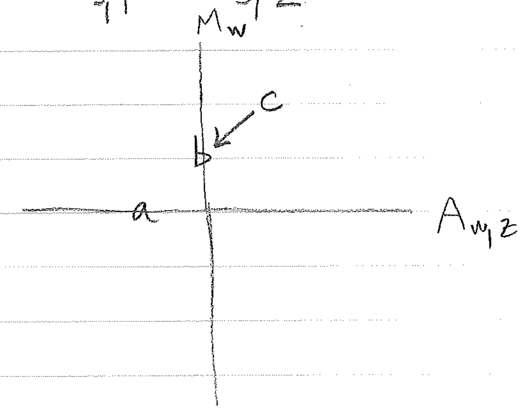
Examples:

(1)



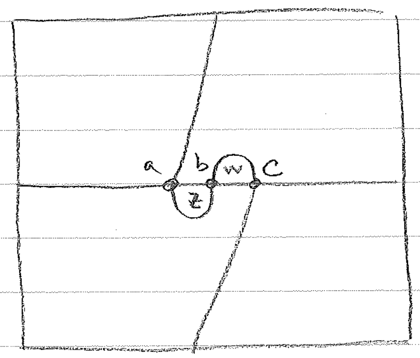
Left-handed trefoil.

$K_{3,1} = T_{3,-2}$

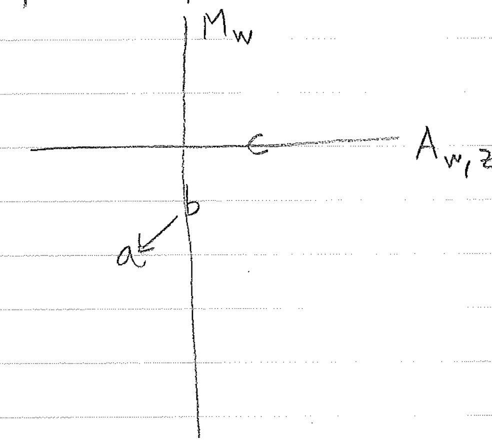


$\tau(T_{3,-2}) = -\underline{1}.$

(2)

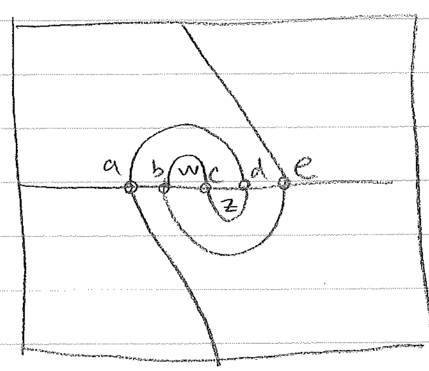


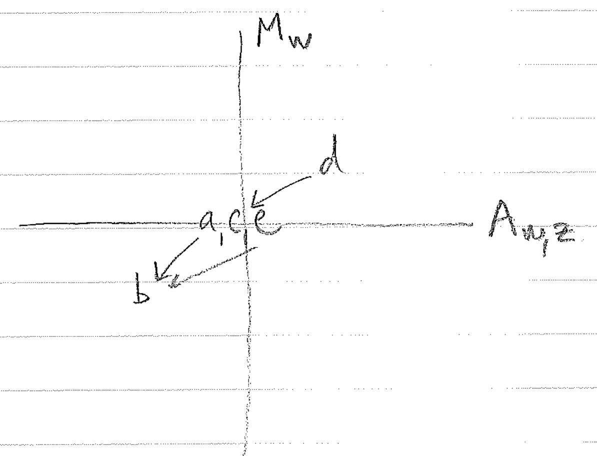
$K_{3,-1} = T_{3,2}$



$\tau(T_{3,2}) = \underline{1}.$

(3)





$\tau(K_{5,2}) = 0$

Careful: Don't be misled by computation for "perfect" knots. (Will give an example on homework of less straightforward τ comp.)