3/26/14

Theorem A: (Oz-Sz) Let $K \subseteq Y^3$ be a nullhomologous knot in a c.c.o. 3-mfd. Y^3 (for simplicity, assume $b_1(Y)=0$). Then the filtered chain homotopy type of the $\mathbb{Z}$-filtered complex $\widehat{CF}(Y,K)$ associated to K is an invariant of K (independent of choices).

induced by Alex. grading

Proof: Delayed.

Corollary: When $Y=S^3$,

$$\tau(K) := \min_{m\in\mathbb{Z}} \left\{ (i_m)_* : H_*(\mathcal{F}_m) \to \widehat{HF}(S^3) \;\middle|\; (i_m)_* \neq 0 \right\}$$

is an invariant of K.

Theorem B: (Oz-Sz) Let $K \subseteq S^3$

(1) $\tau(K\#J) = \tau(K)+\tau(J)$

(2) $\tau(-K) = -\tau(K)$

(3) $\tau(K) \leq g_4(K)$

$\Bigg\}$ τ yields a (surjective) homomorph. $\tau: \mathcal{C} \to \mathbb{Z}$

look @ image of $T_{2,n}$, n odd.

(4) $\tau(T_{p,q}) = \frac{(p-1)(q-1)}{2} \left(= g_4(T_{p,q}) = g(T_{p,q})\right)$.

(NOTE: Rasmussen's $(\frac{1}{2})$s-invariant satisfies all of these properties as well).

In order to understand proof of Theorem B, will need to develop an enhanced version of Heegaard-Floer knot invariant. Enhanced version, $CFK^\infty(Y,K)$, has in it the data of $\widehat{CF}(Y,K)$ but also the data of $\widehat{HF}(Y_r(K))$ for any $r \in \mathbb{Q}$.

$CFK^\infty(Y,K)$ is a $\mathbb{Z}\oplus\mathbb{Z}$-filtered complex. We will argue that its filtered chain homotopy type is an invariant of (Y,K) (Theorem A will be a corollary).

Algeb. structure of

$\underline{CFK^\infty(Y,K)}$: $(\mathbb{Z}\oplus\mathbb{Z})$-filtered complex

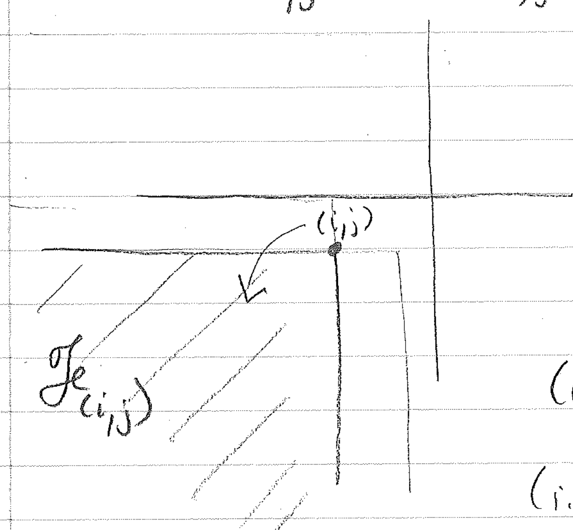
$\mathbb{Z}\oplus\mathbb{Z}$ is partially-ordered, by
$(i,j)\le(i',j')$ if $i\le i'$ and $j\le j'$.

$(\mathbb{Z}\oplus\mathbb{Z})$-filtered complex :

Collection $\{\mathcal{F}_{(i,j)\in\mathbb{Z}\oplus\mathbb{Z}}\}$ of subcomplexes

with $\mathcal{F}_{(i,j)}\subseteq\mathcal{F}_{(i',j')}$ if $(i,j)\le(i',j')$.

(i,j) defines a $(-,-)$ quadrant



$\mathcal{F}_{(i,j)}\subseteq\mathcal{F}_{(i',j')}$ whenever (i,j) quadrant $\subseteq(i',j')$ quadrant (i.e., when $(i,j)\le(i',j')$).

Given $(\Sigma,\vec\alpha,\vec\beta,w,z)$ a 2-pointed Heeg. diag.

Step 1: Define

$CF^\infty(\Sigma,\vec\alpha,\vec\beta,w,z)$: Generators (over $\mathbb{F}$) are triples $[x,i,j]$ w/ $x\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$, $(i,j)\in\mathbb{Z}\oplus\mathbb{Z}$.

3/26/14

(allowing ourselves to count all holom. disks, record w, z inters. counts).

$$\partial^\infty[x,i,j] = \sum_{y\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\{\phi\in\pi_2(x,y)\,|\,\mu(\phi)=1\}} \#(\hat{\mathcal{M}}(\phi))[y, i-n_w(\phi), j-n_z(\phi)]$$

(For this count to be finite, Σ must be strongly $\mathfrak{s}$-admissible for $\mathfrak{s}=\mathfrak{s}_w(x)$)

(see notes on admissibility)

Remark: (1) This complex splits according to $\mathrm{Spin}^c(Y)$ as usual.

(2) A LOT of duplicate information. For each $\mathfrak{s}\in\mathrm{Spin}^c(Y)$, we have $\mathbb{Z}$ isomorphic subcomplexes.

Defn: $CFK^\infty(Y,K) = \mathrm{Span}_{\mathbb{F}}\{[x,i,j] \,|\, A_{w,z}(x) = j-i\}$.

Lemma: $CFK^\infty(Y,K) \subseteq CF^\infty(\Sigma,\vec{\alpha},\vec{\beta},w,z)$ is a subcomplex.

Proof: Suppose $[x, i, i+A_{w,z}(x)] \in CFK^\infty(Y,K)$. We want to see that any $[y,i',j']$ w/ $\langle [y,i',j'], \partial^\infty[x,i,i+A_{w,z}(x)]\rangle$ satisfies
$$A_{w,z}(y) = j'-i'$$

But if $\phi\in\pi_2(x,y)$, we saw that

$$(*)\ A_{w,z}(x) - A_{w,z}(y) = n_z(\phi) - n_w(\phi).$$

Since
$$i' = i - n_w(\phi)$$
$$j' = j - n_z(\phi)$$

$$j'-i' = (j-i) - (n_z(\phi) - n_w(\phi)) = A_{w,z}(x) - (n_z(\phi) - n_w(\phi)) = A_{w,z}(y) \quad \square$$

Note: $CFK^\infty(Y, K)$ is denoted $CFK^\infty(Y, K, \underline{t_0})$ in O–S Knots.

Can define $CFK^\infty(Y, K, \underline{t_m}) := \{[x, i, j] \mid j - i = A_{w,z}(x) + m\}$, which is an isomorphic subcomplex.

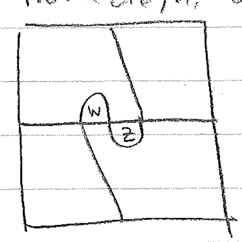
$$CF^\infty(\Sigma, \vec{\alpha}, \vec{\beta}, w, z) = \bigoplus_{m \in \mathbb{Z}} CFK^\infty(Y, K, \underline{t_m}).$$

<u>Claim</u>: $CFK^\infty(Y, K)$ can be given the structure of a module over $\mathbb{F}[U, U^{-1}]$, freely generated by the set $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

<u>Proof</u>: Let $U[x, i, j] := [x, i-1, j-1]$
$(U^{-1}[x, i, j] := [x, i+1, j+1])$.
Then $CFK^\infty(Y, K)$ is freely generated over $\mathbb{F}[U, U^{-1}]$ by

$$\{[x, 0, A_{w,z}(x)] \mid x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta\}.$$

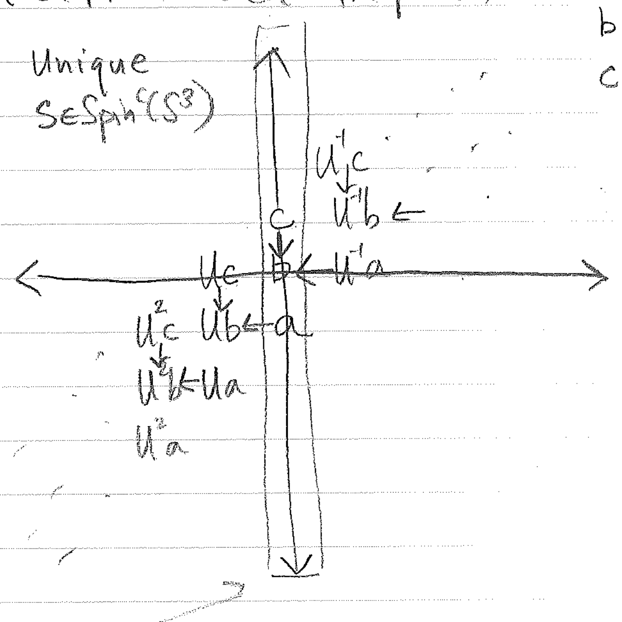
(Also clear from defn. of ∂^∞ that it is U-equivariant). □

<u>Example</u>: 

(Left-handed trefoil).

Unique $s \in Spin^c(S^3)$

$a = [a, 0, -1]$
$b = [b, 0, 0]$
$c = [c, 0, 1]$.



Remark: $\widehat{CF}(S^3, K)$ is a subquotient complex of $CFK^\infty(S^3, K)$