Investigating \( \text{CFK}^\infty(Y, K) \)

**TODAY:**
1. Idea of why \( \mathbb{Z}^2 \)-filtered chain homotopy type is a knot invariant
2. Symmetry \((\Rightarrow \tau(-K) = -\tau(K))\)
3. Behavior under connected sum \((\Rightarrow \tau(K_1 \# K_2) = \tau(K_1) + \tau(K_2))\)

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**TOPLOGICAL LEMMA:** Any two \( \mathbb{Z}^2 \)-pointed Heeg. diagrams \( \Sigma, \Sigma' \) associated to \((Y, K)\) are related by a finite sequence \( \mathcal{R} \)

1. Isotopies of \( \Sigma, \Sigma' \) curves \( \exists \) avoiding \( w, z \)
2. Handleslides among \( \Sigma, \Sigma' \) curves \( \exists \)
3. \((\text{De})/\text{stabilizations}\)

**Proof:** Standard Morse/Cerf theory. Heeg. decom. associated to self-indexing Morse function, Riemannian metric. A generic \( 1 \)-parameter family of Morse functions "compatible with \( K \)" (i.e., for which \( K \) is a union of two distinct flow lines from index 0 to index 3 critical points) intersects only the codim 1 stratum, where we see
- handleslides among \( \Sigma, \Sigma' \) curves
- \((\text{De})/\text{stabilizations}\) \((\text{Death/Birth of a 1-2 pair})\)

**Theorem (Ozsváth-Szabó):** The \( \mathbb{Z}^2 \)-filtered chain homotopy type of \( \text{CFK}^\infty(Y, K) \) is an invariant of \((Y, K)\).

**Proof:** Use chain maps in proof of invariance of \( \text{HF}(Y) \). Since handleslides, \((\text{De})/\text{stabilizations}\) avoid \( w, z \), and we have intersection positivity for holomorphic domains, the maps are \( \mathbb{Z}\mathbb{Z}^2 \)-filtered. Similar for maps associated to changing a. cpx. structure.
Remark: You may wonder about simultaneous \((0,1)\), \((2,3)\) stabilizations, corresponding to introducing a new \((w, z)\) basepoint pair. Will try to write a homework problem about this.

(Can alternatively be accomplished by a \((1,2)\) birth. Also, any subsequent handle slides before de stabilization have analogues in \((1,2)\) setting).

**Corollary**: \(\tau(\varphi, \Theta)\) is an invariant \(\delta_2(Y, K)\) (thought of as map \(\{\text{torsion}(Y)\} \to \mathbb{Z}\)).

**Symmetry**

We will use the following

**Notation/Definition**: Let \(\sigma \in \text{Spin}^c(Y)\) be torsion, so the absolute \(\Phi\) homological grading and absolute \(\mathbb{Z}\) Alexander gradings are well-defined. Then \(\sigma\)
Proposition (cf. Props. 3.8, 3.9 0-S Knots)

Let \((Y, K)\) be a nullhomologous knot, \(\in \text{Spin}^c(Y)\) torsion, and \(-K\) the reverse of the mirror of \(K\) \((K^r = -Y)\).

\(\mathcal{CFK}^\infty(Y, K, \overline{\mathcal{S}})\) is obtained from \(\mathcal{CFK}^\infty(Y, K, \mathcal{S})\) by
- reversing all arrows
- reflecting across line \(y = -x\).

Proof: Let \((\xi, \nu, \beta, \omega, \mathcal{W}, \mathcal{Z})\) be a 2-pointed Heegaard diagram for \((Y, K)\).

Then \(\left(\xi, \beta, \nu, \omega, \mathcal{W}, \mathcal{Z}\right)\) is a 2-pointed H.D. for \((Y, -K)\).

\(\mathcal{CFK}^\infty(\xi, \beta, \nu, \omega, \mathcal{W}, \mathcal{Z})\) has same generators as \(\mathcal{CFK}^\infty(\xi, \nu, \beta, \omega, \mathcal{Z}, \mathcal{W})\) but

- domain \(\phi \in \Pi_2(x, y)\) in 1 corresponds to a (holom.) domain \(\phi' \in \Pi_2(y, x)\) in 2 (reverse the arrows)
- roles of vertical/horizontal axes are reversed (reflect across line of slope -1, i.e., multiply \(x \in \Pi_\alpha \cap \Pi_\beta\) by \(U^{Aw}(\mathcal{S})\)).

Note that:
- \(M_\mathcal{Z}(\overline{x}) = M_\mathcal{W}(U^{Aw}(\mathcal{S})(\overline{x}')) = M_\mathcal{W}(\mathcal{X}') - 2A_{w_1}(\mathcal{X}')\)
- \(A_{Z,W}(\mathcal{X}) = -A_{w_\mathcal{Z}}(\mathcal{X})\)

\(\square\)
Proof (by Heegaard diagram):

Sketch Heegaard diagram. Take two marked Heegaard diagrams. $\CFK^0 (Y, k_2, k_2, k_2)$ is chain homotopy equivalent.

Theorem (O-S, Rasmussen): Let $K, Y, s\in \SpinC(Y)$. If $s$ is spin, $s\in \SpinC(Y)$ torsion.

Example:

Corollary: $\mathcal{I}(-K) = \mathcal{I}(K)$. (Exercise)
For connected sum: replace $z, w'$ with 2 feet of 1-handle.

Corresponds to removing two 3-balls

Identifying common boundary $S^2$'s then connecting the ends of $K, K'$ to $w, w'$ from $z', z'$ to disjoint union

Connected sum.
Make $\Sigma_g \# \Sigma_h$ marked by replacing $\gamma_g = \mu$.

$\gamma_{g+h} := \mu + \mu'$

Replace $\mu = \beta_g$ with $\gamma = \mu + \mu'$

$K \# K'$

feet of $1$-handle.

$(\Sigma \# \Sigma')_\beta = (\alpha_1, \ldots, \alpha_g, \alpha_1', \ldots, \alpha_h')$ \(\beta \# \beta' = (\beta_1, \ldots, \beta_g, \gamma, \beta_1', \ldots, \beta_h')\) \(W_{\Sigma}^\delta\)

Clear that elements of $\Pi_{\alpha} \# \alpha' \cap \Pi_{\beta} \# \beta'$ correspond bijectively to elements of

\[
\left(\Pi_{\alpha} \cap \Pi_{\beta}^{-}\right) \times \left(\Pi_{\alpha}^{-} \cap \Pi_{\beta}\right)
\]

\[
\begin{pmatrix}
\beta_1 & \cdots & \beta_g & \gamma & \beta_1' & \cdots & \beta_h' \\
\vdots & & \ddots & \ddots & \vdots & & \vdots \\
\alpha_1 & \cdots & \alpha_g & \alpha_1' & \cdots & \alpha_h' & 0 \\
0 & & & & 0 & & 0 \\
0 & & & & 0 & & 0 \\
0 & & & & 0 & & 0 \\
\end{pmatrix}
\]

(Nearly block-diagonal)

Summands of determinants are pairs of summands of determinants of blocks.
To see why $\partial^\infty \left( [x, i, j] \otimes [x', i', j'] \right)$

$= \left( \partial^\infty [x, i, j] \right) \otimes [x', i', j'] + [x, i, j] \otimes \partial^\infty [x', i', j']$

See Oz - Sz Knots, Sect. 7 & Oz - Sz 3-man. apps., Section 6

Example: CFK$^\infty (\mathbb{C}T_{2,3})$

Convolve (and symmetrize):

Repeat (Exercise from Jon Hom's talk: $\varepsilon = 0$).