Investigating $CFK^\infty(Y,K)$

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TODAY: ① Idea of why $\mathbb{Z}^2$-filtered chain homotopy type is a knot invariant
② Symmetry ($\Rightarrow \tau(-K) = -\tau(K)$).
③ Behavior under connected sum ($\Rightarrow \tau(K_1 \# K_2) = \tau(K_1)+\tau(K_2)$)

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① TOPOLOGICAL LEMMA: Any two 2-pointed Heeg. diags. Σ, Σ' associated to (Y,K) are related by a finite sequence of

(1) Isotopies of $\vec{\alpha}, \vec{\beta}$ curves
(2) Handleslides among $\{\vec{\alpha}, \vec{\beta}\}$ curves
(3) (De)/stabilizations

} avoiding w, z

Proof: Standard Morse/Cerf theory. Heeg. decomp. associated to self-indexing Morse function, Riemannian metric. A generic 1-parameter family of Morse functions "compatible w/ K" (i.e., for which K is a union of two distinct flowlines from index $0 \to$ index 3 critical points) intersects only the codim 1 singular stratum, where we see

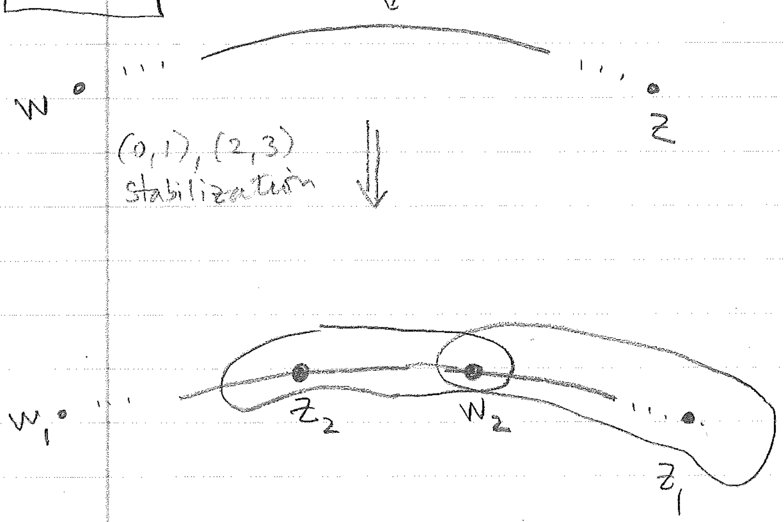
- handleslides among $\{\alpha, \beta\}$ curves
- (de)/stabilizations (Death/Birth of a 1-2 pair).

Theorem (Ozsváth-Szabó): The $\mathbb{Z}^2$-filtered chain homotopy type of $CFK^\infty(Y,K)$ is an invariant of (Y,K).

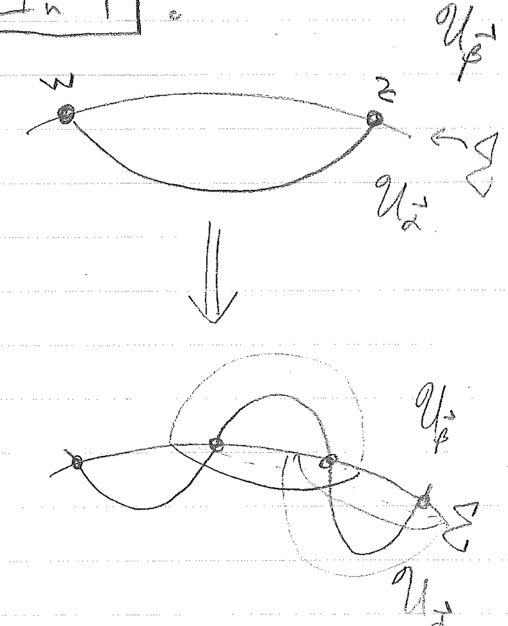
Proof: Use chain maps in proof of invariance of $HF(Y)$. Since handleslides, (de) stabilizations avoid w, z, and we have intersection positivity for holomorphic domains, the maps are $\mathbb{Z}\oplus\mathbb{Z}$-filtered. Similar for maps associated to changing a.cpx. structure. □

Remark: You may wonder about simultaneous $(0,1)$, $(2,3)$ stabilizations, corresponding to introducing a new (w,z) basepoint pair. Will try to write a homework problem about this.

$\boxed{\text{On } \Sigma}$:



$\boxed{\text{In } Y}$:



(Can alternatively be accomplished by a (1-2) birth. Also, any subsequent handleslides before destabilization have analogues in (1-2) setting).

COROLLARY: $\tau(Y,K)$ is an invariant of (Y,K) (thought of as map $\{\text{Torsion Spin}^c(Y)\} \to \mathbb{Z}$).

② SYMMETRY

We will use the following

Notation/definition: Let $\mathfrak{s} \in \text{Spin}^c(Y)$ be torsion, so the absolute $\mathbb{Q}$ homological grading and absolute $\mathbb{Z}$ Alexander gradings are well-defined. Then:

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Proposition (cf. Props. 3.8, 3.9 O-S Knots)

Let (Y, K) be a nullhomologous knot, $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ torsion, and $-K$ the reverse of the mirror of K ($K^r \subseteq -Y$).

$CFK^\infty(Y, \overline{K}, \overline{\mathfrak{s}})$ is $\mathbb{Z}^2$-homotopy-equivalent to the complex obtained from $CFK^\infty(Y, K, \mathfrak{s})$

by
- reversing all arrows
- reflecting across line $y = -x$.

Proof: Let $(\Sigma, \vec{\alpha}, \vec{\beta}, w, z)$ be a 2-pointed Heegaard diagram for (Y, K).

Then $(\underbrace{-\Sigma, \vec{\beta}, \vec{\alpha}}_{-Y}, \underbrace{z, w}_{K^r})$ is a 2-pointed H.D. for $(Y, -K)$.

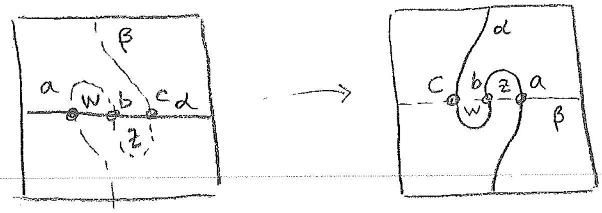
① $CFK^\infty(-\Sigma, \vec{\beta}, \vec{\alpha}, z, w)$ has same generators as ② $CFK^\infty(\Sigma, \vec{\alpha}, \vec{\beta}, w, z)$ but

- (holomorphic) domain $\phi \in \pi_2(x, y)$ in ① corresponds to a (holom.) domain $\phi' \in \pi_2(y, x)$ in ② (reverse the arrows)
- Roles of vertical / horizontal axes are reversed (reflect across line of slope -1, i.e. multiply $\vec{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ by $U^{A_{w,z}(\vec{x})}$).

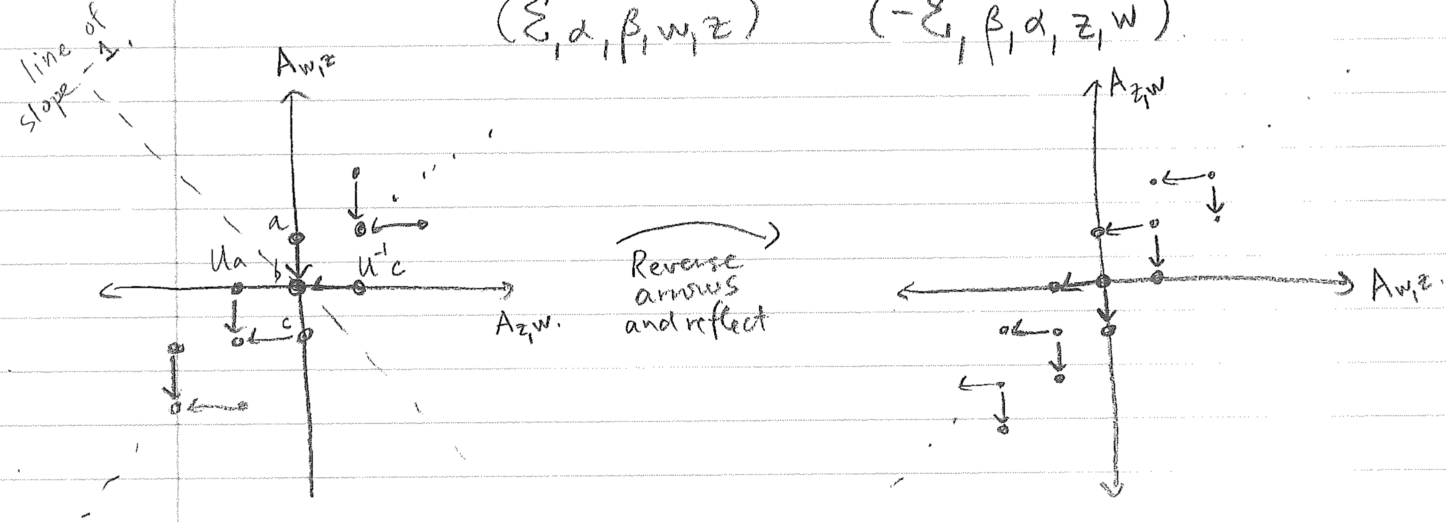
Note that:
- $$M_z(\vec{x}) = M_w(U^{A_{w,z}(\vec{x})}(\vec{x})) = M_w(\vec{x}) - 2A_{w,z}(\vec{x})$$
- $$A_{z,w}(\vec{x}) = -A_{w,z}(\vec{x})$$

$\square$

Example:



$(\Sigma, \alpha, \beta, w, z)$ $\quad$ $(-\Sigma, \beta, \alpha, z, w)$



Corollary: $\tau(-K) = -\tau(K)$. (Exercise)

Theorem (O-S, Rasmussen): Let $K_1 \subseteq Y_1$, $K_2 \subseteq Y_2$, $s_1 \in Spin^c(Y_1)$, $s_2 \in Spin^c(Y_2)$ torsion,

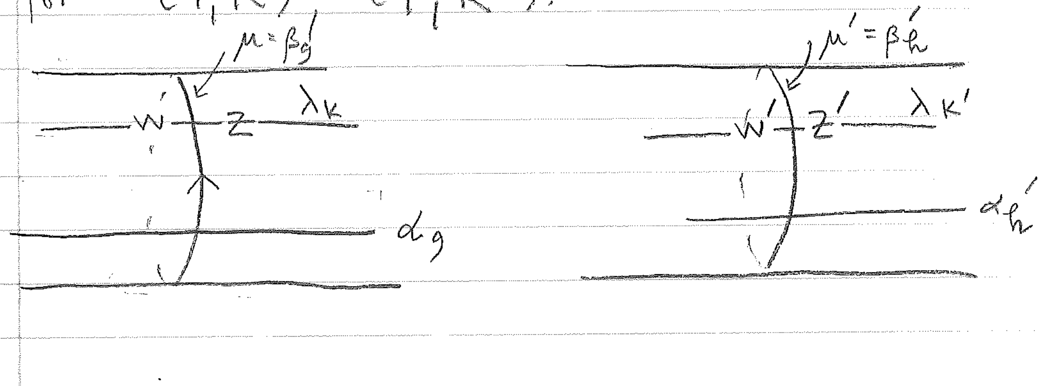
$CFK^\infty(Y_1 \# Y_2, K_1 \# K_2; s_1 \# s_2)$ is $\mathbb{Z} \oplus \mathbb{Z}$ filtered chain homotopy equivalent to

$$CFK^\infty(Y_1, K_1; s_1) \otimes_{\mathbb{F}[U^{\pm}]} CFK^\infty(Y_2, K_2; s_2).$$

Proof Sketch: (by Heegaard diagram): Take two marked Heegaard diagrams

$(\Sigma_g, \vec{\alpha}, \vec{\beta}, w, z)$, $(\Sigma'_{h}, \vec{\alpha}', \vec{\beta}', w', z')$

for (Y, K), (Y', K').



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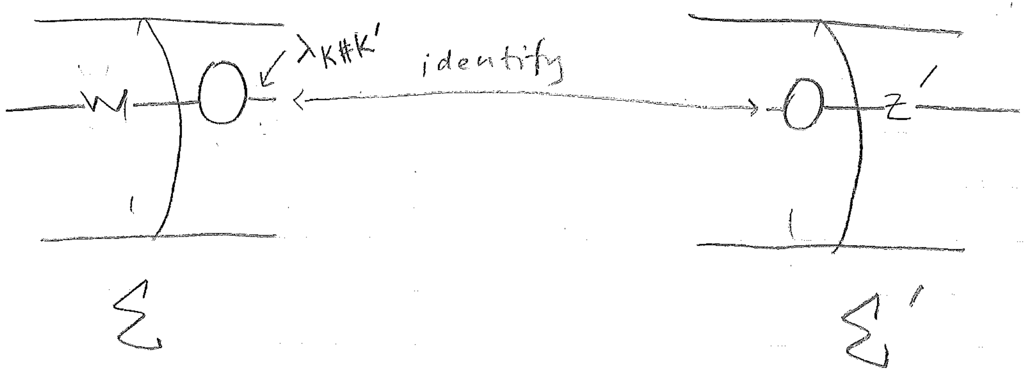
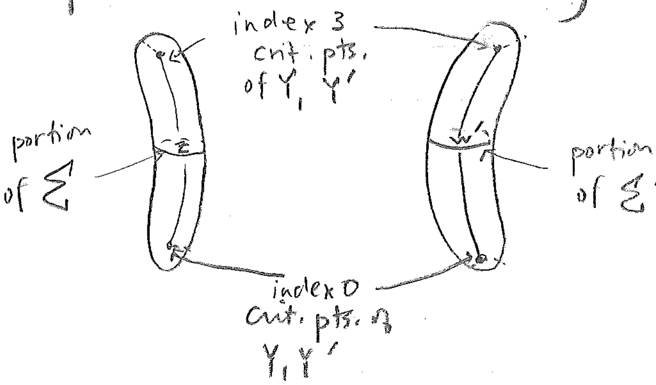
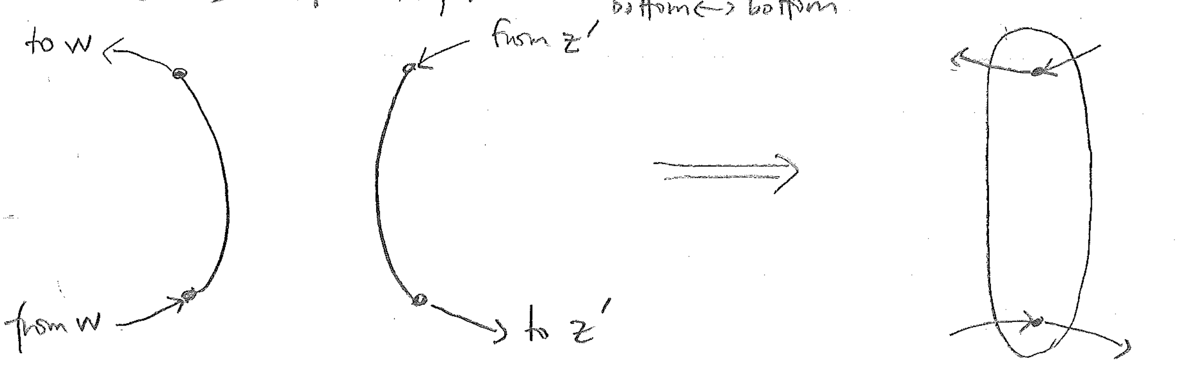
For connected sum: replace z, w' with 2 feet of 1-handle.



Corresponds to removing two 3-balls



identifying common boundary S^2's, then connecting the ends of K, K'
top $\longleftrightarrow$ top'
bottom $\longleftrightarrow$ bottom

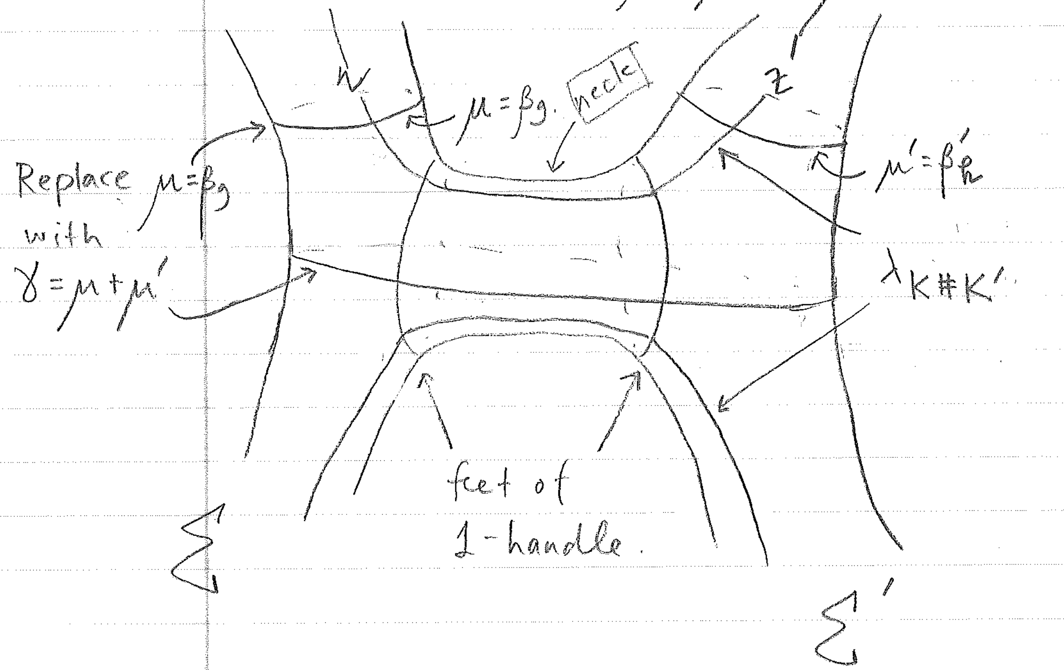


Disjoint union

Connected sum.

Make $\overbrace{\Sigma_g \# \Sigma'_h}^{\text{genus } g+h}$ <u>marked</u> by replacing $\beta_g = \mu$ with

$$\gamma_{g+h} := \mu + \mu'$$



$$\left(\Sigma \# \Sigma',\ \vec{\alpha}\#\vec{\alpha}' = (\alpha_1, \dots, \alpha_g, \alpha'_1, \dots, \alpha'_h),\ \vec{\beta}\#\vec{\beta}' = (\beta_1, \dots, \beta_{g-1}, \underset{\substack{\uparrow \\ \text{replaces } \mu}}{\gamma}, \beta'_1, \dots, \underset{\substack{\uparrow \\ \text{meridian}}}{\beta'_h}),\ \underset{\substack{\uparrow \\ \text{can now travel across neck.}}}{w, z'}\right)$$

Clear that elements of $\mathbb{T}_{\vec{\alpha}\#\vec{\alpha}'} \cap \mathbb{T}_{\vec{\beta}\#\vec{\beta}'}$ correspond bijectively to elements of

$$(\mathbb{T}_{\vec{\alpha}} \cap \mathbb{T}_{\vec{\beta}}) \times (\mathbb{T}_{\vec{\alpha}'} \cap \mathbb{T}_{\vec{\beta}'}).$$

$$\begin{array}{c|c}
 & \beta_1 \ \cdots \ \beta_{g-1} \quad \gamma \quad \beta'_1 \ \cdots \ \beta'_h \\
\begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_g \\ \alpha'_1 \\ \vdots \\ \alpha'_h \end{array} &
\left(\begin{array}{c|c|c|c}
\mathbb{T}_{\vec{\alpha}_0} \cap \mathbb{T}_{\vec{\beta}_0} & 0 & 0 & 0 \\
\hline
* & q' & * & 0 \\
\hline
0 & 0 & \mathbb{T}_{\vec{\alpha}'_0} \cap \mathbb{T}_{\vec{\beta}'_0} & 0 \\
\hline
* & q'' & * & q
\end{array}\right)
\end{array}$$

(<u>Nearly</u> block-diagonal)

Summands of determinant are pairs of summands of determinants of blocks

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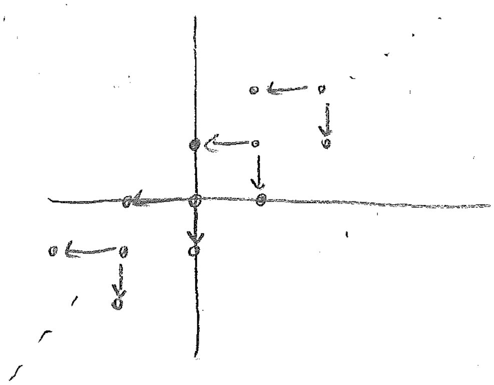
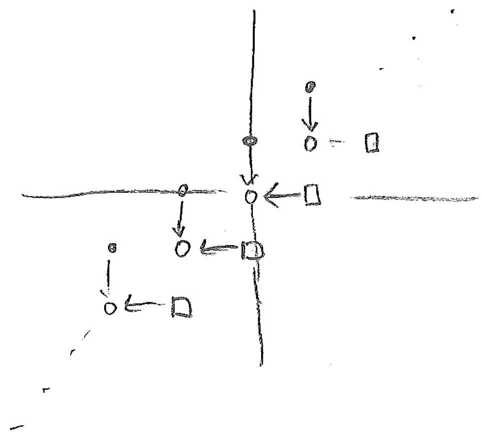
To see why $\partial^\infty\big(\overbrace{[x,i,j]\otimes[x',i',j']}^{[(x,x'),\,i+i',\,j+j']}\big)$

$$=\left(\partial^\infty[x,i,j]\right)\otimes[x',i',j'] + [x,i,j]\otimes\left(\partial^\infty[x',i',j']\right)$$

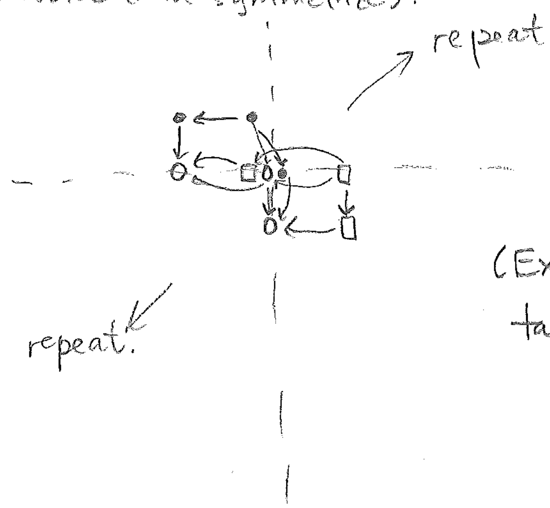
see Oz-Sz Knots, Sect. 7 &
Oz-Sz 3-man. apps., Section 6

Example: $CFK^\infty(T_{2,3})$ $\qquad$ $CFK^\infty(-T_{2,3})$





Convolve (and symmetrize):



(Exercise from Jen Hom's talk: $\varepsilon = 0$).