p. 1

4/14/14

Last time: Proved ① $\tau(-K) = -\tau(K)$
② $\tau(K_1 \# K_2) = \tau(K_1) + \tau(K_2)$.

Today: ① Prove $|\tau(K)| \leq g_4(K)$ (modulo an argument using adjunction inequality)
② Introduce a construction of Honda-Kazez-Mátic/Torisu in prep. for next topic.

For simplicity of exposition, assume $K \subseteq S^3$.

$|\tau(K)| \leq g_4(K)$

We may assume WLOG (by replacing K w/ $-K$ and using ①) that $\tau(K) \geq 0$.

Will need this later in argument

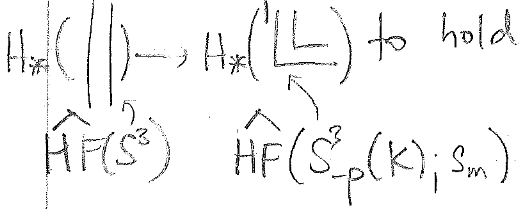
We may assume WLOG (by replacing K w/ $K \# T_{2,3}$ if necessary) that $1 \leq g_4(K)$.

(Note: Seems like circular reasoning, but there are classical obstructions to $T_{2,3}$ being slice, so we know $g_4(T_{2,3}) = 1$.)

Remark: The above replacement will increase τ by 1 (by ②) and g_4 by 1 (since $g_4(K \# T_{2,3}) \leq g_4(K) + g_4(T_{2,3}) = 1$, but $g_4(K \# T_{2,3}) \neq 0$, since this would imply $g_4(T_{2,3}) = 0$.), so proving

$$\underset{\tau(K)+1}{\underset{\shortparallel}{\tau(K \# T_{2,3})}} \leq \underset{g_4(K)+1}{\underset{\shortparallel}{g_4(K \# T_{2,3})}} \implies \tau(K) \leq g_4(K)$$

Choose $p \in \mathbb{Z}^+$ sufficiently large for Theorem about cobordism maps from CFK^∞ "hook maps":



to hold.

and suppose $m \in \mathbb{Z} < \tau(K)$.

We argued before that the map

$$\widehat{HF}(S^3) \longrightarrow \widehat{HF}(S^3_{-p}(K), s_m)$$

is therefore nontrivial.

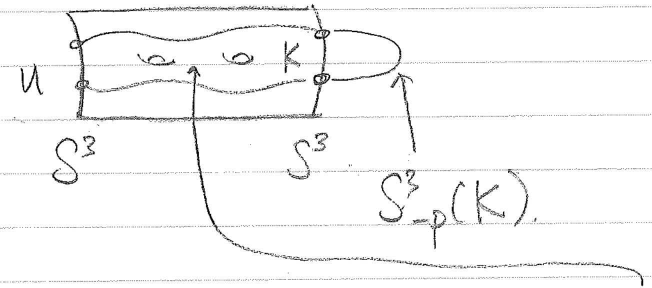
Recall: s_m is the restriction to $S^3_{-p}(K)$ of the Spin^c structure $t_m \in \text{Spin}^c(W_{-p})$

2-handle addition along K

with $\langle c_1(t_m), [\hat{F}] \rangle + \underbrace{[\hat{F}]\cdot[\hat{F}]}_{-p} = 2m$

capped off
surface bounded by K,
smoothly imbedded
in B^4

Note: By removing the nhd. of a pt. in B^4 We can think of this cobordism as a composition of the identity map & this nontrivial map.



Choose the pt. to be on $F \subseteq B^4$ and note that Oz-Sz have proved (Theorem 1.1, O-S 4-mfds.) that the map is an invariant of the pair (4-mfd. cobordism, spin^c structure)

So we are free to factor the map as a composition:

$$S^3 \longrightarrow \partial N_{-p} \longrightarrow S^3_{-p}(K)$$

where $N_{-p} = \nu(F)$
$= D^2$ bundle with Euler # $-p$.

But O-S have calculated $\widehat{HF}(S^3) \to \widehat{HF}(\partial N_{-p})$ using LES and a direct calculation of $\widehat{HF}(\partial N_0)$ (adjunction inequality: supported in only certain spin^c structures).

Will try to return to this later: uses a particular Heegaard diagram compatible w/ F.

Conclusion: $F_{N_{-p}, t_m} : \widehat{HF}(S^3) \to \widehat{HF}(\partial N_{-p}, s_m)$

Restrictions of spin^c structures defined before.

is trivial unless

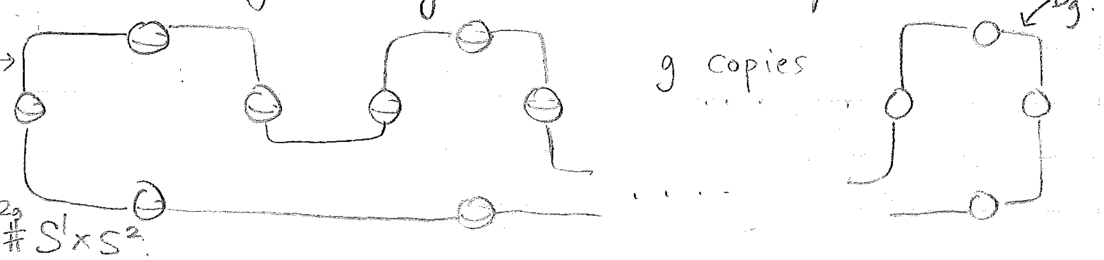
$$\underbrace{\langle c_1(t_m), [\hat{F}] \rangle + [\hat{F}]\cdot[\hat{F}]}_{2m} \leq 2(g(\hat{F})-1).$$

$F_{N_{-p}, t_{\tau(K)-1}}$ non trivial $\Longrightarrow 2(\tau(K)-1) \leq 2(g(\hat{F})-1)$

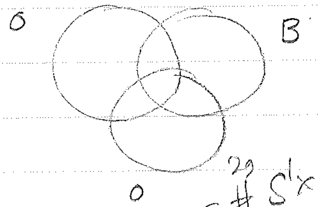
$$\Longrightarrow \tau(K) \leq g(\hat{F}).$$

Possible way to understand this map directly: N_{-p} has the following concrete description

$-p$ surgery on this (knot Floer simple) knot in $\#^{2g} S^1 \times S^2$.



This is $-p$ surgery on $B_g = \overbrace{(B\#\ldots\#B)}^{g \text{ copies}} \subseteq \overset{2g}{\#}(S^1\times S^2)$, where B is the "Borromean knot":



$\subseteq \overset{2g}{\#} S^1\times S^2$

Claim (check): B_g is a fibred knot with trivial monodromy.

It is proved in O-S Knots, Prop. 9.2, that $\mathrm{rk}(\widehat{HFK}(B)) = \mathrm{rk}(\widehat{HF}(\overset{2}{\#} S^1\times S^2)) \Rightarrow B$ is a "knot Floer simple" knot in $\overset{2}{\#} S^1\times S^2$. (uses a LES on $\widehat{HFK}$).

But we have a "Honda-Kazez-Mátić" Heegaard diagram for $B \subseteq \overset{2}{\#} S^1\times S^2$ constructed as follows (see H-K-M "On the contact class...". See also Sec. 5.2, Ozbagci-Stipsicz: a construction of Torisu)

Let $L \subseteq Y^3$ be a fibered link with fiber S and monodromy h (satisfies $h|_{\partial S} = id_{\partial S}$).

↳ Explicitly:
$$\underset{\uparrow}{Y^3} = \left(S\times[0,1] \Big/ \begin{array}{l} (x,1)\sim(h(x),0) \text{ for } x\in S \\ (y,t)\sim(y,t') \text{ for } y\in\partial S \end{array} \right)$$

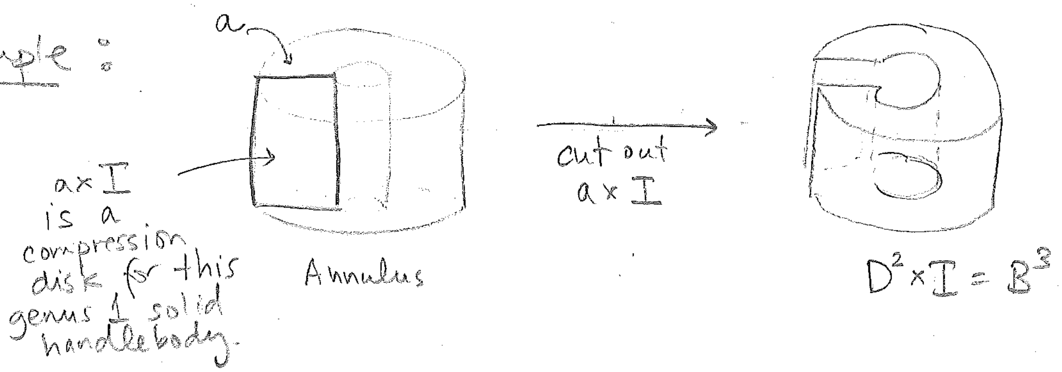
"Open book decomposition" of Y

<u>Remark</u>: $S\times[0,\tfrac{1}{2}]$ and $S\times[\tfrac{1}{2},1]$ are handlebodies:

The existence of such a disjoint collection of compressing disks is what defines a handlebody.

A basis of "cutting arcs" on S specifies a collection of decomposing <u>compression disks</u>. Cut along them to obtain $D^2 \times I = B^3$

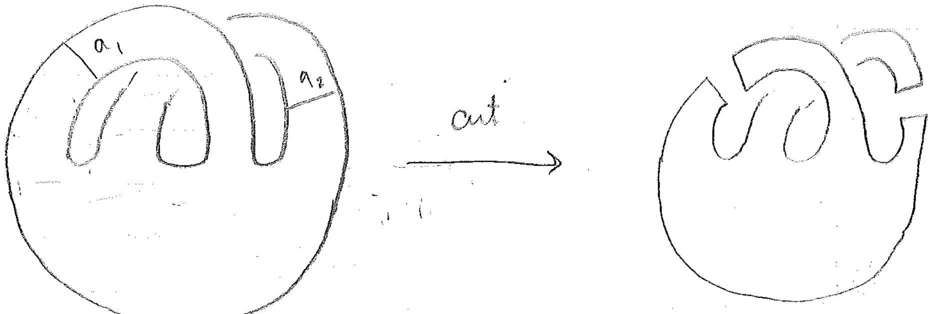
<u>Example</u>:



$a \times I$ is a compression disk for this genus 1 solid handlebody.

Annulus

$D^2 \times I = B^3$



1-punctured T^2 (thickened)

D^2 (thickened)

$\Longrightarrow$ HKM / Torisu Heegaard diagram for (S, h): (open book) (choose a collection $a_1, \ldots, a_n$ of cutting arcs on S).

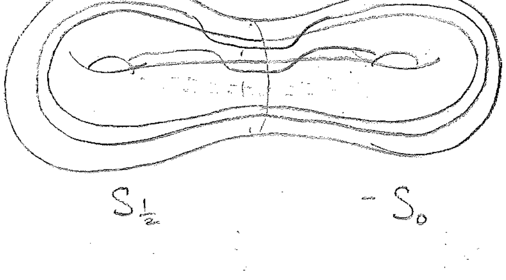
$$\Sigma = \underset{\cup}{S_{\frac{1}{2}}} \cup - \underset{\cup}{S_0}$$

$$\alpha_i = a_i \cup a_i$$

$$\beta_i = a_i' \cup h(a_i')$$

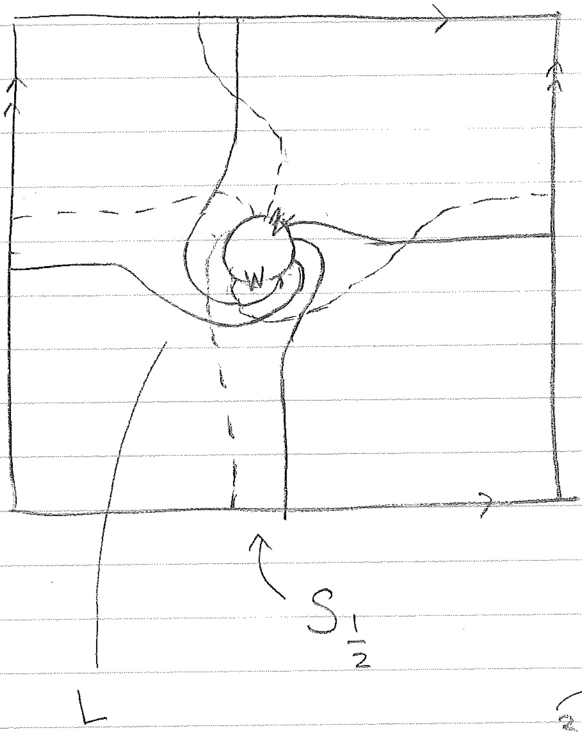
Perturbed along boundary to achieve transverse intersection

<u>Example</u>: HKM/T Heegaard diagram associated to $B \leq \#^2 S^1 \times S^2$;



If $\overset{\text{fibered}}{L}$ has l components (i.e., S has l boundary components), then we can obtain a $2l$-pointed Heegaard diagram for $L \subseteq Y^3$ by "winding".

In above example:



Exercise(?): Compute $CFK^\infty(\overbrace{\#^2 S^1\times S^2}^{Y}, B)$ from this Heegaard diagram. Verify directly that

$$\underbrace{\widehat{HF}(Y)}_{\substack{\text{Supported}\\ \text{in a single}\\ \text{Spin}^c\text{ structure } \mathfrak{s}\\ \text{(torsion)}}} \to \widehat{HF}(Y_{-p}(B), \mathfrak{s}_m) \quad \text{is trivial unless } m \le 2\underbrace{g(S)}_{1} - 1.$$

Prove also for B_g by using Künneth formula.