Last time: Proved \( \tau(-K) = -\tau(K) \)
\( \tau(K_1 \# K_2) = \tau(K_1) + \tau(K_2) \).

Today: 1. Prove \( |\tau(K)| \leq g_4(K) \) (modulo an argument using adjunction inequality).
2. Introduce a construction of Honda–Kazez–Matić/Torisu in prep. for next topic.

For simplicity of exposition, assume \( K \cong S^3 \).

\[ |\tau(K)| \leq g_4(K) \]

We may assume WLDG- (by replacing \( K \) with \( -K \) and using 1) that \( \tau(K) \geq 0 \).

We may assume WLDG- (by replacing \( K \) with \( K \# T_2,3 \), if necessary) that \( 1 \leq g_4(K) \).

(Note: Seems like circular reasoning, but there are classical obstructions to \( T_2,3 \) being slice, so we know \( g_4(T_2,3) = 1 \).

Remark: The above replacement will increase \( \tau \) by 1 (by 2) and \( g_4 \) by 1.
(Since \( g_4(K \# T_2,3) \leq g_4(K) + g_4(T_2,3) = 1 \), but \( g_4(K \# T_2,3) \neq 0 \), since this would imply \( g_4(T_2,3) = 0 \)), so proving \( \tau(K) \cong T_2,3 \) implies \( g_4(K) = 1 \).

Choose \( p \in \mathbb{Z}^+ \) sufficiently large for Theorem about bordism maps from \( CFK^* \) "hook maps":

\[ H^*_\#(\mathcal{L}) \to H^*_\#(\mathcal{L}) \]

\[ \hat{HF}(S^3) \to \hat{HF}(S^3(p(K); \mathfrak{m})) \]

and suppose \( m \in \mathbb{Z}^+ < \tau(K) \).
We argued before that the map
\[ \hat{HF}(S^3) \rightarrow \hat{HF}(S^3_{-p}(K), S^3_m) \]
is therefore nontrivial.

Recall \( S^3_m \) is the restriction to \( S^3_{-p}(K) \)
of the \( \text{Spin}^c \) structure \( t_m \in \text{Spin}^c(W_{-p}) \)
with \[ \langle c_1(t_m), [\hat{F}] \rangle + [\hat{F}] \cdot [\hat{F}] = 2m \]
capped off
surface bounded by \( K \),
smoothly imbedded in \( B^4 \)

By remuring the nhd. of a pt. in \( B^4 \)

Note: We can think of this cobordism as a composition of the identity map & this nontrivial map.

Choose the pt. to be on \( F \subset B^4 \)
and note that Oz-Sz have proved (Theorem 1.1, O-S 4-mfds.) that the map is an invariant of the \( \text{pair}(4\text{-mfd, cobordism, } \text{Spin}^c \text{ structure}) \)

So we are free to factor the map as a composition.
\[ S^3 \to \#N_p \to S^3(K) \]

where \( N_p = J(F) = D^2 \) bundle with Euler \( \# -p \).

But O-S have calculated \( \hat{HF}(S^3) \to \hat{HF}(\#N_p) \) using LES and a direct calculation \( \hat{HF}(\#N_0) \) (adjunction inequality supported in only certain spin \( \text{c} \) structures).

\( \hat{F} \) Will try to return to this later: uses a particular Heegaard diagram compatible w/ \( F \).

Conclusion: \( F_{N_p, t_m} \circ \hat{HF}(S^3) \to \hat{HF}(\#N_p, S_m) \)

is trivial unless

\[
\langle c_1(t_m), [\hat{F}] \rangle + [\hat{F}] \cdot [\hat{F}] \leq 2(g(\hat{F}) - 1)
\]

\( F_{N_p, t_m} \) non-trivial \( \Rightarrow 2(2m - 1) \leq 2(g(\hat{F}) - 1) \) \( \Rightarrow t(K) \leq g(\hat{F}) \).

Possible way to understand this map directly: \( N_p \) has the following concrete description:

\[ -p \text{ surgery on this (knot Floer (knot in \( \# S^1 \times S^2 \) \))} \]

\[ g \text{ copies} \]

\[ B_0 \]
This is $p$-surgery on $(B\#\ldots\#B)\cong 2^g(\Sigma^1 \times S^2)$, where $B$ is the "Borromean knot":

![Diagram of Borromean knots]

Claim (check): $B_g$ is a fibred knot with trivial monodromy.

It is proved in O-S Knots, Prop. 9.2, that $\text{rk}(\text{HFK}(B)) = \text{rk}(\text{HF}(\# S^1 \times S^2)) \Rightarrow B$ is a "knot Floer simple" knot in $\# S^1 \times S^2$, (uses a LES on HFK).

But we have a "Honda-Kazez-Matic" Heegaard diagram for $B \cong \# S^1 \times S^2$ constructed as follows (see H-K-M "On the contact class...". See also Sec. 5.2, Ozbagci-Stipsicz: a construction of Tansu.)

Let $L \subseteq Y^3$ be a fibred link with fiber $S$ and monodromy $\phi$ (satisfies $\phi|_{S^2} = \text{id}_{S^2}$).

$\downarrow$ Explicitly: $Y^2 = (S \times [0,1]) / (x,0) \sim (\phi(x),0)$ for $x \in S$

$\downarrow$ (y,t) $\sim$ (y, t') for y $\in S$

"Open book decomposition" of $Y$

Remark: $S \times [0, \frac{1}{2}]$ and $S \times [\frac{1}{2}, 1]$ are handlebodies.
The existence of such a disjoint collection of compressing disks is what defines a handlebody.

A basis of "cutting arcs" on $S$ specifies a collection of decomposing compression disks. Cut along them to obtain $D^2 \times I = B^3$.

Example:

$A \times I$ is a compression disk for this genus $4$ solid handlebody. $\amalg$ Annulus

$1$-punctured $T^2$ (thickened)

$D^2$ (thickened)

$\Rightarrow$ HKM/Toricu Heegaard diagram for $(S_1, h)$: (choose a collection $a_1, \ldots, a_n$ of cutting arcs on $S$).

$\Sigma = S_{1/2} \cup \cdots \cup S_0$

$\alpha_i = a_i \cup a_i^{-1}$

$\beta_i = a_i \cup h(a_i)$

Example: A Heegaard diagram associated to $B = \# S \times S$: $S_{1/2} - S_0$
If \( L \) has \( l \) components (i.e. \( S \) has \( l \) boundary components), then we can obtain a \( 2l \)-pointed Heegaard diagram for \( L \subseteq Y^3 \) by "winding".

In above example:

Exercise(9): Compute \( \text{CFK}^\infty(\#S^1 \times S^2, B) \) from this Heegaard diagram. Verify directly that

\[
\text{HF}(Y) \to \text{HF}(Y_p(B), S_m) \quad \text{is trivial unless} \quad m \leq 2g(S) - 1.
\]

Prove also for \( B_g \) by using Künneth formula.