Class next week: W, F instead of M, W.

$K \subseteq Y^3$ (nullhomologous, $b_1(Y)=0$). p. 1

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Recall: Last time we discussed how to see HF of "sufficiently large" integer surgery in terms of subquotient complexes of $CFK^\infty(Y,K)$.

Today: ◦ See how to describe HF cobordism maps

$(p \in \mathbb{Z}^+, p >> 0)$ $\quad \widehat{HF}(Y) \longrightarrow \widehat{HF}(Y_{-p}(K))$

$$\widehat{HF}(Y) \longleftarrow \widehat{HF}(Y_p(K))$$

in terms of data in $CFK^\infty(Y,K)$.

◦ Characterize $\tau(K)$ in terms of maps above.
◦ Give idea of proof that $\widehat{HF}(Y_{-p}(K))$ can be seen from $CFK^\infty(Y,K)$

Theorem: (Oz-Sz. See Thm. 4.1, 4.4 "Knots", See also Thm. 2.3 "Knot Floer homology & Integer Surgeries")
Fix $(\Sigma, \vec{\alpha}, \vec{\beta}, w, z)$ an admissible Heeg. diag. for (Y,K). $\exists\, N \in \mathbb{Z}^+$ s.t. $\forall\ p \geq N$ we have chain maps Ψ, Φ_{-p}, $F_{-p,m}$, $G_{-p,m}$ fitting into a commutative diag:

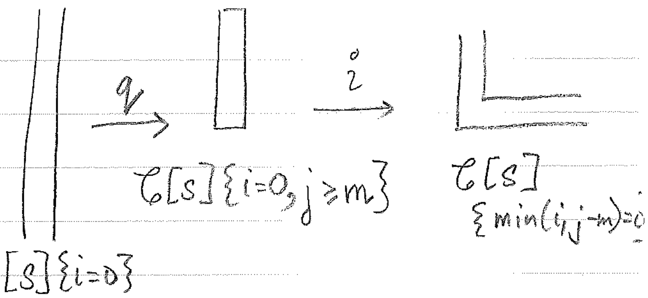
$$\begin{tikzcd} \mathcal{C}[s]\{i=0\} \arrow[r, "\Psi"] \arrow[d, "F_{-p,m}"'] \arrow[dr, phantom, "\circlearrowright"] & \widehat{CF}(Y,s) \arrow[d, "G_{-p,m}"] \\ \mathcal{C}[s]\{\min(i,j-m)=0\} \arrow[r, "\Phi_{-p,m}"'] & \widehat{CF}(Y_{-p}(K), s_m) \end{tikzcd}$$

(for each $-\frac{p}{2} < m < \frac{p}{2}$)

such that ◦ Ψ, $\Phi_{-p,m}$ induce isomorphisms in homology

◦ $G_{-p,m}$ is defined by counting hol. triangles

$\circ\ F_{-p,m} = \overset{\text{include}}{\mathring{i}} \circ \overset{\text{quotient}}{q}$

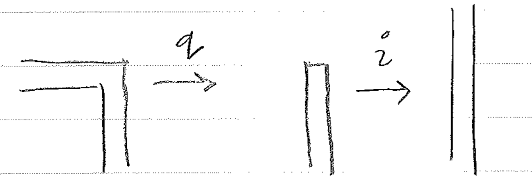


$$\mathcal{C}[s]\{i=0\} \xrightarrow{q} \mathcal{C}[s]\{i=0, j \geq m\} \xrightarrow{\mathring{i}} \mathcal{C}[s]\{\min(i,j-m)=0\}$$

Analogously, we have chain maps Ψ, $\Phi_{p,m}$, $F_{p,m}$, $G_{p,m}$

$$\begin{tikzcd}
\mathcal{C}[s]\{i=0\} \arrow[r, "\Psi"] & \widehat{CF}(Y, \mathfrak{s}) \\
\mathcal{C}[\mathfrak{s}]\{\max\{i, j-m\}=0\} \arrow[u, "F_{p,m}"] \arrow[r, "\Phi_{p,m}"'] & \widehat{CF}(Y_p(K), \mathfrak{s}_m) \arrow[u, "G_{p,m}"']
\end{tikzcd}$$

where $F_{p,m} = \mathring{i} \circ q$



__CONCLUSION__: Maps on HF homology induced by cobordisms (sufficiently large +/− integer surgeries) can be read off from $CFK^\infty(Y,K)$.

Indeed, $\tau(\xi, \text{—})$ also has an interpretation in terms of these maps!

RESTRICT TO $K \subset S^3$:

__Proposition__: (3.1 in O-S "4-ball genus") If $m < \tau(K)$, then

$$(G_{-p,m})_* : \widehat{HF}(S^3) \longrightarrow \widehat{HF}(S^3_{-p}(K), \mathfrak{s}_m)$$

is nontrivial for $p \in \mathbb{Z}^{>>0}$.

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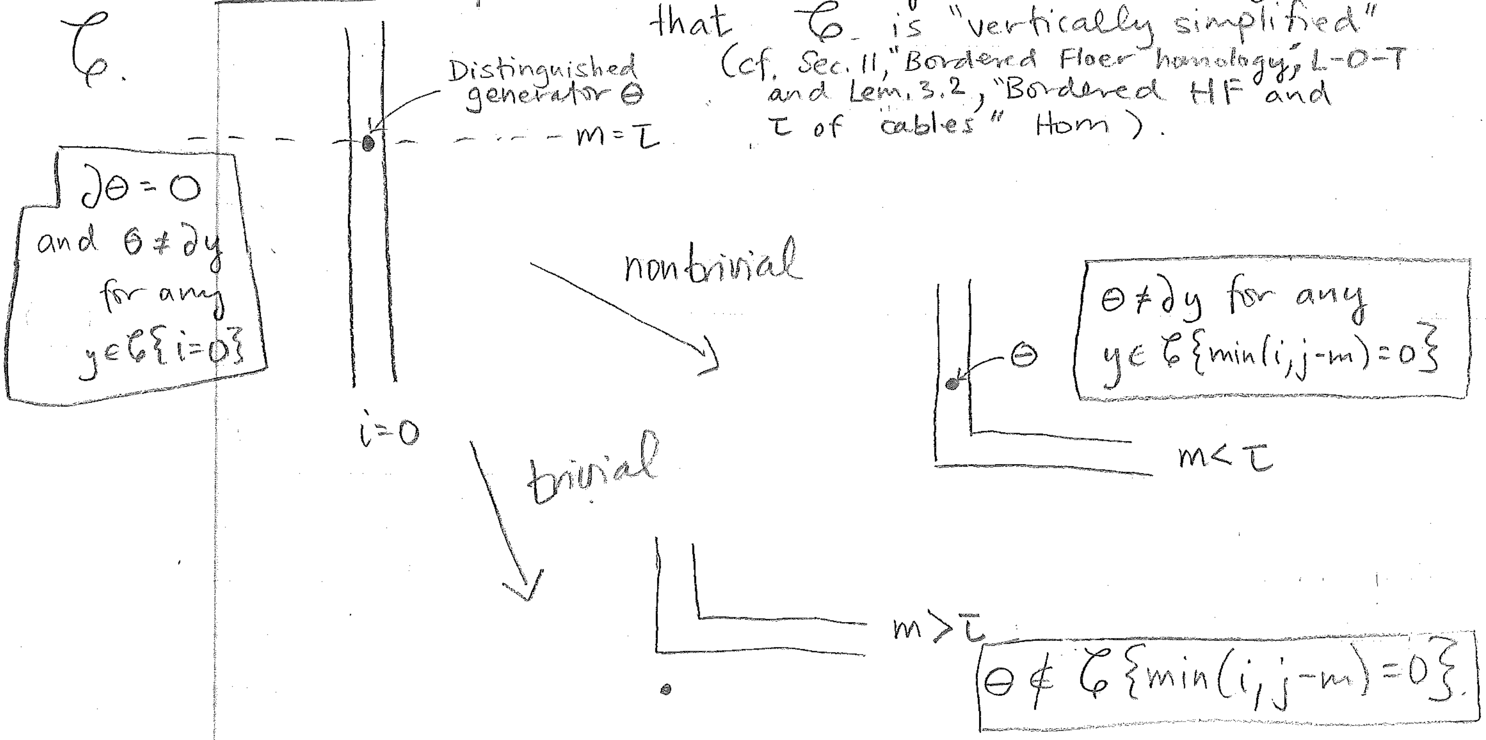
If $m > \tau(K)$, then $(G_{-p,m})_*$ is trivial for $p \in \mathbb{Z}^{>>0}$.

So we can think of $\tau(K)$ as the "transition" value for map induced by 2-handle cobordisms (with sufficiently negative framing) to be trivial/non.

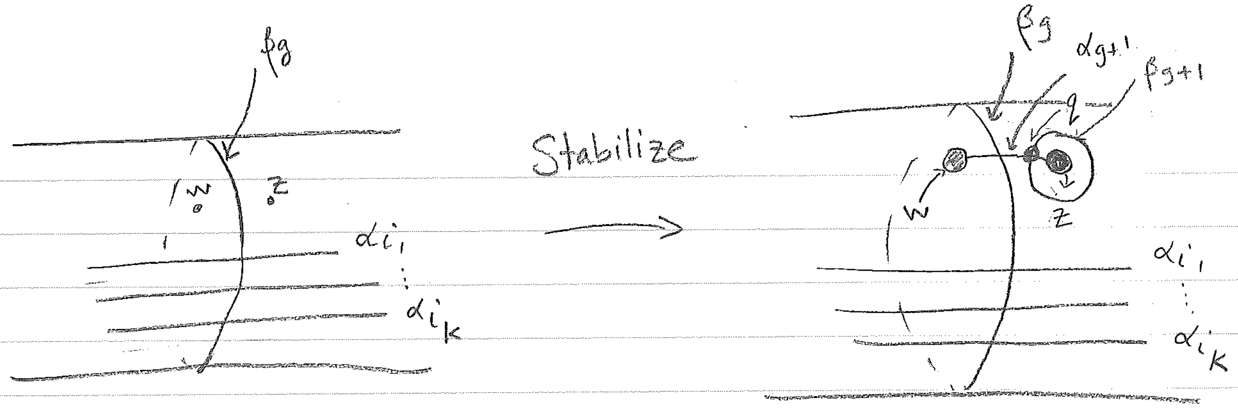
<u>Proof of Proposition</u>: Arrange by $\mathbb{Z}^2$-filtered change of basis that $\mathcal{C}$ is "vertically simplified" (cf. Sec. 11, "Bordered Floer homology," L-O-T and Lem. 3.2, "Bordered HF and τ of cables" Hom).



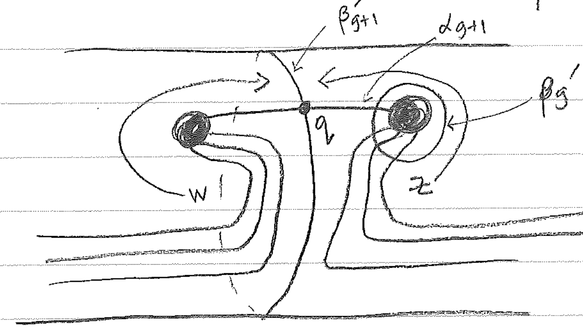
□.

<u>IDEA OF PROOF OF THEOREM</u>: We've fixed a <u>marked</u> 2-pointed Heegaard diagram for (Y, K):
$(\Sigma, \vec{\alpha}, \vec{\beta} = \vec{\beta_0} \cup \mu, w, z)$.

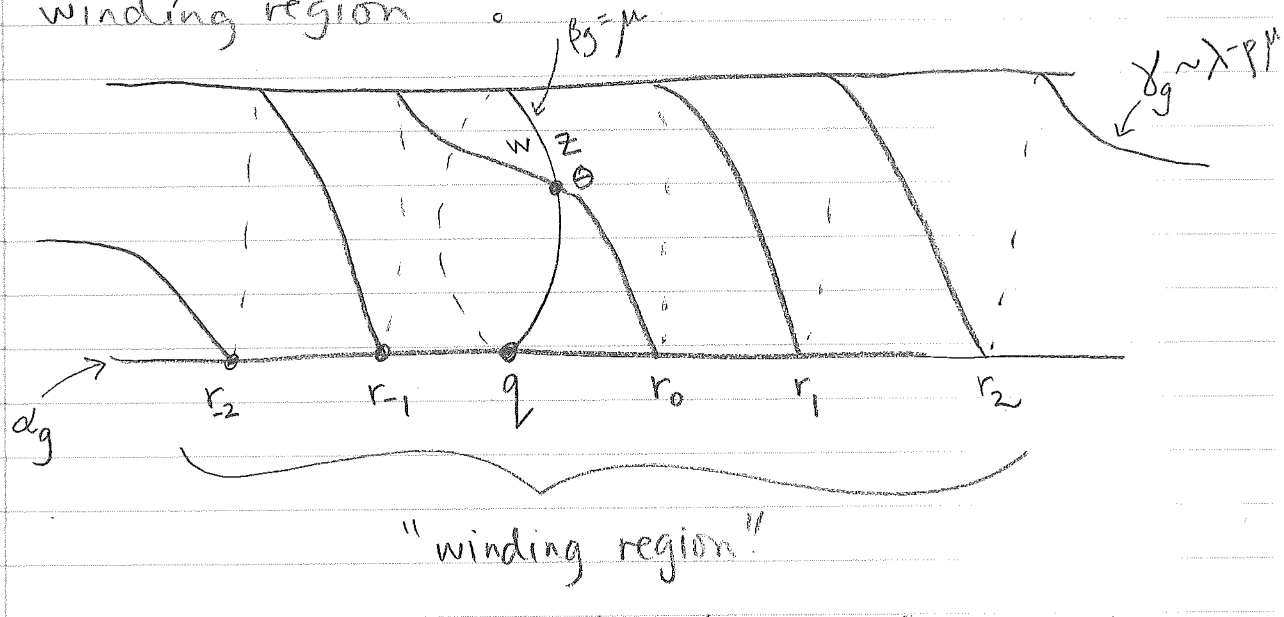
* We may assume that β_g intersects α_g in a single intersection point, q, and intersects no other $\vec{\alpha}$'s.
(cf. Lemma 3.4, Rasmussen's thesis: Stabilize and handleslide



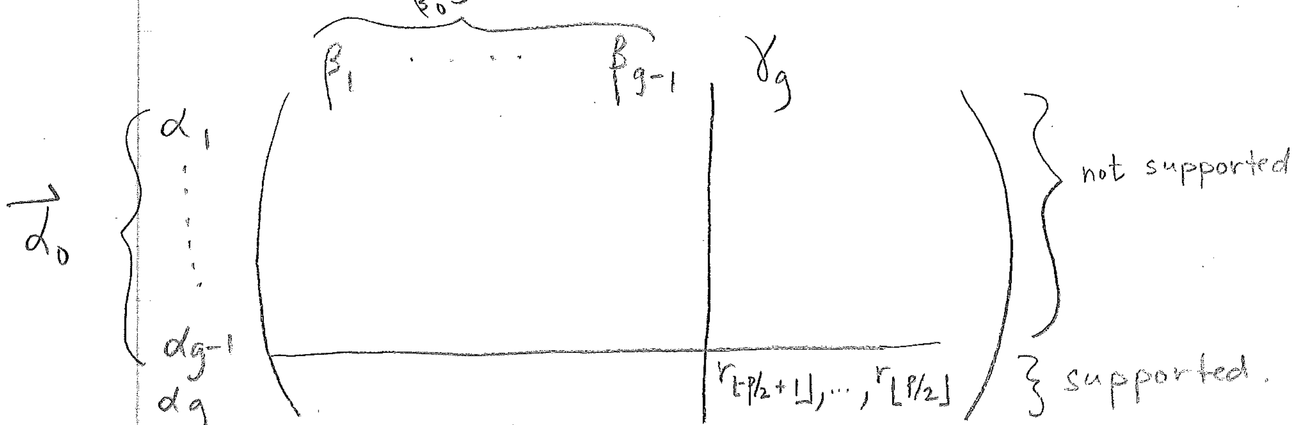
Note that β_{g+1} is a meridian for K and intersects α_{g+1} in a single point, q. Alternatively, rename $\beta_{g+1}' = \beta_g$, $\beta_g' = \beta_{g+1}$ and handleslide $\alpha_{i_1} \cdots \alpha_{i_k}$ off of β_{g+1}':



We can now obtain a Heegaard diagram $(\Sigma, \vec{\alpha} = \vec{\alpha}_0 \cup \alpha_g, \vec{\gamma} = \vec{\beta}_0 \cup \gamma_g, w)$ for $Y_{-p}(K)$ for any $p \in \mathbb{Z}^+$ by replacing $\beta_g = \mu$ with γ_g, a (nearby) smoothing of $\lambda - p\mu$. By convention, place w in middle of "winding region":



Notice that there are two types of intersection points in $\mathbb{T}_{\vec{\alpha}} \cap \mathbb{T}_{\vec{\gamma}}$, those $\left\{ \begin{matrix} \text{supported} \\ \text{not supported} \end{matrix} \right\}$ in winding region :



Int. pts. of $\mathbb{T}_{\vec{\alpha}} \cap \mathbb{T}_{\vec{\gamma}}$ supported in winding region are of the form

$$(\vec{x}_0, r_m) \quad \text{for} \quad x_0 \in \mathbb{T}_{\vec{\alpha}_0} \cap \mathbb{T}_{\vec{\beta}_0}$$

Moreover, given $(\vec{x}_0, q) \in \mathbb{T}_{\vec{\alpha}} \cap \mathbb{T}_{\vec{\beta}}$

q ← lone int. btw. α_g, β_g

$$s_w(\vec{x}_0, r_m) = \underbrace{s_{m - A_{w,z}(x)}}_{} \in Spin^c(Y_{-p}(K))$$

← restriction to $Y_{-p}(K)$ of $t_m \in Spin^c(W_{-p})$

- extends $s_w(\vec{x}_0, q)$
- $\langle c_1(t_m), [\hat{F}] \rangle - p = 2(m - A_{w,z}(x))$.

(indeed, we see an obvious small triangle ψ "from" $q \to r_m$, going through Θ representing a term in the cobordism map

$$\widehat{HF}(Y, s) \longrightarrow \widehat{HF}(Y_{-p}(K), s_m))$$