TODAY: Prove $\tau(T_{r,s}) = \frac{1}{2} (r-1)(s-1)$ (Here $(r,s) = 1$, so $T_{r,s}$ is a knot.)

Follows from:

Theorem (O-S, "Lens Space Surgeries"): Let $K \subseteq \mathbb{S}^3$ satisfy the property that $\exists p \in \mathbb{Z}^+"$ sufficiently large $: p > 2g(K)-1$ such that $\mathbb{S}^3(K) \cong L(p,q)$ is a lens space.

Then $\tau(K) = d$
\[= \text{degree of (symmetrized) Alexander polynomial}.\]

(Indeed, we will see that any knot admitting a positive lens space surgery has very special form.)

Recall: $L(p,q)$ is $-p/q$ surgery on $U \subseteq \mathbb{S}^3$ along curve $\mathcal{C} = -p\mu + q\lambda$.

From corresponding genus 1 Heegaard diagram, we have

\[\text{rk}(\hat{C}F(L(p,q))) = p,\]

But $H_1(L(p,q)) \cong \mathbb{Z}/p\mathbb{Z}$, so
\[|\text{Spin}^c(L(p,q))| = p\]

$\Rightarrow \text{rk}(\hat{H}F(L(p,q))) = p_i$ (Rank 1 in each Spin$^c$ structure).
Note: In general, if $Y$ is a $\Omega HS^3$ ($H_* (Y; \Omega) \approx H_* (S^3; \Omega)$)

$$\text{rk} \left( \hat{H}F (Y, s) \right) > 1 \quad \forall \ s \in \text{Spin}^c (Y)$$

(categorifies "Turaev torsion" of $Y$).

**Definition:** A $\Omega HS^3$ $Y$ with

$$\text{rk} \left( \hat{H}F (Y, s) \right) = 1 \quad \forall \ s \in \text{Spin}^c (Y)$$

is called an $L$-space ("HF lens space")

(***) Relationship $\text{CFK}^{\infty} (S^1, K) \leftrightarrow \hat{H}F$ of large surgeries on $K$

gives strong restrictions on $K \in S^3$

admitting lens space (and, more generally $L$-space) surgeries.

**Some notation:** Let

$$Z_m = \text{"max-hook"} \ @ \ j = m = \bigcap \{ \max (i, j - m) = 0 \}$$

$X_m = \text{horiz. piece} = \bigcap \{ i \leq 0, j = m \}$

$Y_m = \text{vert. piece} = \bigcap \{ i = 0, j \leq m \}$
\[ X_m = \bigcap_{\ell < 0, j = m}^\mathcal{C} \quad X^0 = \bigcap_{\ell = 0, j = m}^\mathcal{C} \quad Y_m = \bigcap_{\ell = 0, j < m}^\mathcal{C} \quad Y^0 = \bigcap_{\ell = 0, j = m}^\mathcal{C} \]

Note: 1. Since \( \text{CFK}^{\kappa}(K) \) is \( U \)-equivariant, \( UX_m = X_{m-1} \). Allows us to set up a "reverse induction."  2. \( H_*(X^0) \) is, by definition, the homology of \( \hat{HF}(K; m) \), associated graded of \( CF(S^3, K) \).

Assume \( p \in \mathbb{Z}^+ \) so that

"Base case" \( \hat{HF}(S^3_p(K); S_m) = \mathbb{H}_*(Z_m) \)

Observation: For \( m \in \mathbb{Z}^+ \) sufficiently large, \( H_*(X^-) = 0 \).

Algebraic Lemma (3.1 of D-S "Lens Space Surgeries"): Suppose \( H_*(Z_m) \cong \mathbb{F} \quad \forall m \in \mathbb{Z} \). Suppose \( H_*(X^-) = 0 \). Then one of the following occurs:

1. \( H_*(X^0) = 0 \), and hence \( H_*(UX_m = X_{m-1}) \cong 0 \).
2. \( H_*(X^0) \cong \mathbb{F} \), and hence \( H_*(UX_m = X_{m-1}) \cong \mathbb{F} \) and \( H_*(Y_m) = 0 \).

Idea of Proof: Use LES's assoc. to SES's

\[ 0 \to X_m \to X_m \to X^0_m \to 0 \]
\[ 0 \to Y_m \to Y_m \to Y^0_m \to 0 \]

along with subquotient complex \( Z_m \cup X^- \)

\[ Z_m \]

\[ X^{-}_{m-1} \]

\[ \square \]
Corollary: Let \( M = \max \{ m \in \mathbb{Z} \mid H_*(X_m) = \hat{HFK}(K, m) \neq 0 \} \).
Then \( M = \tau(K) \).

Proof: \( H_*(X_m) = 0 \) by reverse inductive hypothesis. Since \( H_*(X_m) \neq 0 \), we are in case 2 of Lemma
So \( H_*(Y_-) = 0 \) hence
\[
(i_{M-1})^* H_*(Y_- = \mathcal{L}_{M-1}) \rightarrow \hat{HF}(S^3) \text{ is } 0.
\Rightarrow M \leq \tau(K)
\]
But \( \tau(K) > M \) since
\[
H_*(\mathcal{L}_M) \rightarrow H_*(\mathcal{L}_{M+1}) \rightarrow \ldots \rightarrow \hat{HF}(S^3)
\]
are all isomorphisms (since \( H_*(\mathcal{L}_n/\mathcal{L}_n) = 0 \quad \forall \ n \geq M \)).

(\text{So if } H_*(\mathcal{L}_n) \rightarrow \hat{HF}(S^3) \text{ surjective, we would conclude that } H_*(\mathcal{L}_{n-1}) \rightarrow \hat{HF}(S^3) \text{ already surjective).}

\]
So now we've proved that since \( T_{1,s} \) has a positive lens space (L-space) surgery
\( \tau(T_{1,s}) = M \) \text{ max Alexander grading where } \hat{HFK}(T_{1,s}) \text{ is supported.}

But we also have

Theorem (O-S, Rasmussen): Symmetrized Alexander polynomial
\[
\chi(\hat{HFK}(S^2, K)) = \Delta(K) \quad \sum \frac{\varepsilon(a)}{d} (-1)^d \text{rk}(\hat{HFK}(K, a)) \tau^a
\]
and (cf. Lickorish) the unsymmetrized Alex. poly. of $T_{p, q}$
\[
\frac{(T^r - 1)(T - 1)}{(T - 1)(T^s - 1)}
\]

\[
= \prod_{\xi \neq \xi_0} (T - \xi)
\]

$s_0 = 1,$
$s_0 \neq 1,$
$s_0 \neq 1.$

Its degree is $rs + 1 - r - s = (r-1)(s-1).$
\Rightarrow degree of symmetrized A.P. is $\frac{1}{2}(r-1)(s-1).$
\Rightarrow $M \geq \frac{1}{2}(r-1)(s-1).$

So once again we have
\[
\frac{1}{2}(r-1)(s-1) \leq g(T_{p, q}) \leq g_4(T_{p, q}) \leq g(T_{p, q}) \leq \frac{1}{2}(r-1)(s-1). \square
\]

Remark: Torus knots have genus 1 doubly pointed Heegaard diagrams, so $\tau$ can be computed directly (Good exercise).

Isotop $\beta$ and place $w/z$ so that $\delta_a \cup \delta_b$ is a slope $\frac{3}{2}$ curve on $T_{p, q}.$
Q: Where'd the complex geometry go?
A: Recall that $Tr_s (r, s \in \mathbb{Z}^+)$ is the closure of a positive braid.

We also know Milnor's construction:

$$Tr_s = V_f \cap S^3_\varepsilon$$

where

$$f(z, w) = z^r + w^s$$

is a polynomial.

$$V_f = \{ (z, w) \in \mathbb{C}^2 | f(z, w) = 0 \}$$

These two facts are closely related!

Building on Milnor's idea:

(Lee Rudolph, "Algebraic Functions & Closed Braids")

Suppose

$$f(z, w) = f_0(z)w^n + f_1(z)w^{n-1} + \ldots + f_n(z) \in \mathbb{C}[z, w]$$

As long as:

1. $f_0(z) \neq 0$
2. $f(z, w)$ is square-free
3. $f(z, w)$ has no factors of the form $z-c$

$\Rightarrow f(z, w)$ is an $n$-valued algebraic function of $z$.

$\Rightarrow B = \{ z_0 \in \mathbb{C} | \exists$ fewer than $n$ distinct solutions for $f(z_0, w) \}$ is a finite set.

Choose $\gamma \subset \mathbb{C} - B$.

$\uparrow$ simple closed curve.
For each \( z_0 \in \gamma \), we have \( n \) distinct values of \( w \in \mathbb{C} \) satisfying \( f(z_0, w) = 0 \).

(\( \gamma \) compact \( \Rightarrow \) \( \exists D_R \subseteq \mathbb{C} \) such that all roots of \( f(z_0, w) \) for all values of \( z_0 \) are contained in \( D_R \)).

\[ \Rightarrow \text{Get an \( n \)-stand braid } \sigma = (x) D_R \]

\[ \uparrow \quad \uparrow \]

\[ \text{solid torus.} \]

These two solid tori glue along \( S' \times S' \) boundary to form an imbedded (PL) copy of \( S^3 \subseteq \mathbb{C}^2 \).

(We shall see that the corresponding closed braid naturally transverse link w/ respect to \( \xi_{ST} \), the standard tight contact structure on \( S^3 \)).

Rudolph, Boileau-orevkov. Give a topological characterization of which braids can appear via this construction (Quasipositive braids).

It follows from Milnor conjecture, this work that it is precisely this set of closed braids for which \( r = \frac{5}{3} \).