(p.1)

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<u>TODAY</u>: Prove $\tau(T_{r,s}) = \frac{1}{2}(r-1)(s-1)$ (Here $(r,s)=1$, so $T_{r,s}$ is a <u>knot</u>.)

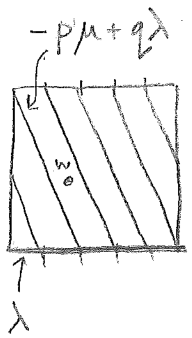
$\boxed{\text{Follows from}}$:

<u>Theorem</u> (O–S, "Lens Space Surgeries"): Let $K \subseteq S^3$ satisfy the property that $\exists\ p \in \mathbb{Z}^+$ ("sufficiently large": $p \geq 2g(K)-1$) such that
$S^3_p(K) \cong L(p,q)$ is a <u>lens space</u>.

Then $\tau(K) = d$
=: degree of (symmetrized) Alexander polynomial.

(Indeed, we will see that the knot Floer homology of any knot admitting a positive lens space surgery has very special form.)
(Louise Moser 1971: Positive torus knots admit positive lens space surgeries.)

<u>Recall</u>: $L(p,q)$ is $-p/q$ surgery on $\mathcal{U} \subseteq S^3$
Along curve $-p\mu + q\lambda$



From corresponding genus 1 Heegaard diagram, we have

$$rk(\widehat{CF}(L(p,q))) = p$$

But $H_1(L(p,q)) \cong \mathbb{Z}/p\mathbb{Z}$, so
$$|Spin^c(L(p,q))| = p$$

(Generators are all in different $\mathcal{E}$-classes $\Rightarrow$ $Spin^c$ classes).

$\Longrightarrow rk(\widehat{HF}(L(p,q))) = p$ (Rank 1 in each $Spin^c$ structure).

Note: In general, if Y is a $\mathbb{Q}HS^3$ ($H_*(Y;\mathbb{Q}) \cong H_*(S^3;\mathbb{Q})$)

$$\mathrm{rk}(\widehat{HF}(Y,\mathfrak{s})) \geq 1 \quad \forall \quad \mathfrak{s} \in \mathrm{Spin}^c(Y)$$

(categorifies "Turaev torsion" of Y).

<u>Definition</u>: A $\mathbb{Q}HS^3$ Y with

$$\mathrm{rk}(\widehat{HF}(Y,\mathfrak{s})) = 1 \quad \forall \quad \mathfrak{s} \in \mathrm{Spin}^c(Y)$$

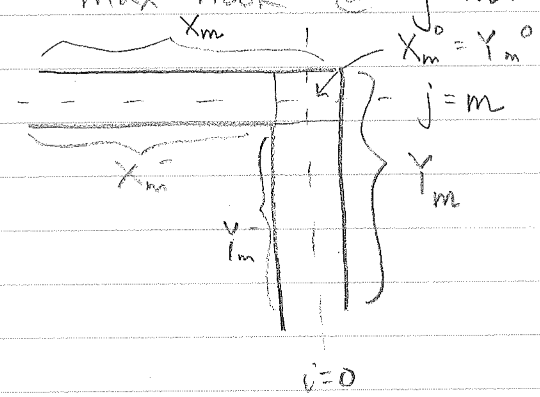
is called an <u>L-space</u> ("HF lens space")

(**) Relationship $CFK^\infty(S^3,K) \longleftrightarrow \widehat{HF}$ of large surgeries on K

gives strong restrictions on $K \subseteq S^3$ admitting lens space (and, more generally L-space) surgeries.

<u>Some notation</u>: Let

$$Z_m = \text{"max-hook" @ } j=m. = C\{\max(i, j-m) = 0\}.$$



X_m = horiz. piece = $C\{i \leq 0, j = m\}$

Y_m = vert. piece = $C\{i = 0, j \leq m\}$

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$X_m^- = \mathcal{C}\{i<0, j=m\}$ $\quad X_m^0 = \mathcal{C}\{i=0, j=m\}$

$Y_m^- = \mathcal{C}\{i=0, j<m\}$ $\quad Y_m^0 =$

Note: ① Since $CFK^\infty(K)$ is $\mathcal{U}$-equivariant, $\mathcal{U}X_m = X_{m-1}$. Allows us to set up "reverse induction"

② $H_*(X_m^0)$ is, by definition $\widehat{HFK}(K;m)$ (homology of associated graded of $\widehat{CF}(S^3,K)$).

Assume $p \in \mathbb{Z}^+ \gg 0$ so that

"Base case" $\quad \widehat{HF}(S^3_p(K); \mathfrak{s}_m) \cong H_*(Z_m)$

<u>Observation</u>: For $m \in \mathbb{Z}^+$ sufficiently large, $H_*(X_m^-) = 0$

<u>Algebraic Lemma</u> (3.1 of O-S "Lens Space Surgeries"):

"Inductive Step": Suppose $H_*(Z_m) \cong \mathbb{F}$ $\forall$ $m \in \mathbb{Z}$.
Suppose $H_*(X_m^-) \cong 0$. Then one of the following occurs:

① $H_*(X_m^0) \cong 0$, and hence $H_*(\mathcal{U}X_m = X_{m-1}) \cong 0$.

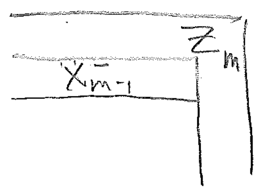
② $H_*(X_m^0) \cong \mathbb{F}$, and hence $H_*(\mathcal{U}X_m = X_{m-1}) \cong \mathbb{F}$.

and $H_*(Y_m^-) \cong 0$.

<u>Idea of Proof</u>: Use LES's assoc. to SES's

$$0 \to X_m^- \to X_m \to X_m^0 \to 0$$
$$0 \to Y_m^- \to Y_m \to Y_m^0 \to 0$$

along with subquotient complex $Z_m \cup X_{m-1}^-$



□.

K be a knot admitting positive integral L-space surgery and suppose

Corollary: Let $M = \max\{m \in \mathbb{Z} \mid H_*(X_m^\circ) = \widehat{HFK}(K;m) \neq 0\}$. Then $M = \tau(K)$.

Proof: $H_*(X_M^-) = 0$ by reverse inductive hypoth.

Since $H_*(X_M^\circ) \neq 0$, we are in case ② of Lemma

So $H_*(Y_M^-) = 0$, hence

$(i_{M-1})_*: H_*(Y_M^- = \mathcal{F}_{M-1}) \longrightarrow \widehat{HF}(S^3)$ is 0.

$\Longrightarrow M \leq \tau(K)$

But $\tau(K) \geq M$ since

$$H_*(\mathcal{F}_M) \longrightarrow H_*(\mathcal{F}_{M+1}) \to \cdots \to \widehat{HF}(S^3)$$

are all isomorphisms (since $H_*(\mathcal{F}_{n+1}/\mathcal{F}_n) \cong 0$ $\forall\, n \geq M$).

(So if $H_*(\mathcal{F}_n) \longrightarrow \widehat{HF}(S^3)$ surjective, $n \geq M+1$, we would conclude that $H_*(\mathcal{F}_{n-1}) \to \widehat{HF}(S^3)$ already surjective.) $\square$

So now we've proved that since $T_{r,s}$ has a positive lens space (L-space) surgery,

$\tau(T_{r,s}) = M$ ← max Alexander grading where $\widehat{HFK}(T_{r,s})$ is supported.

But we also have

Theorem (O-S, Rasmussen):

$$\chi(\widehat{HFK}(S^3, K)) = \Delta(K)$$

Symmetrized Alexander polynomial of K.

$$= \sum_{d \in \mathbb{Z}} \sum_{a \in \mathbb{Z}} (-1)^d \, rk(\widehat{HFK}_d(K;a))\, T^a$$

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and (cf. Lickorish) the unsymmetrized Alex. poly. of $T_{p,q}$:

$$\frac{(T^{rs}-1)(T-1)}{(T^r-1)(T^s-1)}$$

$$= \prod_{\substack{\xi^{rs}=1,\\ \xi^r\neq 1,\\ \xi^s \neq 1}} (T-\xi)$$

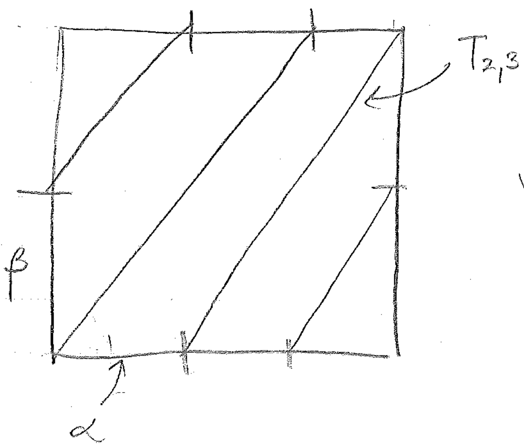
Its degree is $rs+1-r-s = (r-1)(s-1)$.

$\Longrightarrow$ degree of symmetrized A.P. is $\frac{1}{2}(r-1)(s-1)$.
$\Longrightarrow$ $M \geqslant \frac{1}{2}(r-1)(s-1)$.

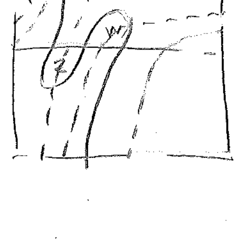
So once again we have

$$\tfrac{1}{2}(r-1)(s-1) \leq \tau(T_{r,s}) \leq g_4(T_{r,s}) \leq g(T_{r,s}) \leq \tfrac{1}{2}(r-1)(s-1) \quad \square.$$

Remark: Torus knots have genus 1 doubly-pointed Heegaard diagrams, so τ can be computed directly (Good exercise).



Isotop β and place w, z so that $\gamma_a \cup \gamma_b$ is a slope $\frac{3}{2}$ curve on torus:



Q: Where'd the complex geometry go?
A: Recall that $T_{r,s}$ $(r,s \in \mathbb{Z}^+)$ is the closure of a positive braid.

We also know <u>Milnor's construction</u>:

$$T_{r,s} = V_f \cap S^3_\varepsilon \quad \text{where}$$

$f(z,w) = z^r + w^s$ is a polynomial.

$V_f = \{(z,w) \in \mathbb{C}^2 \mid f(z,w) = 0\}$

These two facts are closely related!

<u>Building on Milnor's idea</u>:

(Lee Rudolph, "Algebraic Functions & Closed Braids")

Suppose $f(z,w) = f_0(z)w^n + f_1(z)w^{n-1} + \dots + f_n(z) \in \mathbb{C}[z,w]$

As long as: ① $f_0(z) \neq 0$
② $f(z,w)$ is square-free
③ $f(z,w)$ has no factors of the form $z-c$.

$\Rightarrow$ $f(z,w)$ is an <u>n-valued algebraic function of z</u>.

$\Rightarrow$ $B = \{z_0 \in \mathbb{C} \mid \exists$ fewer than n distinct solutions for $f(z_0,w)\}$ is a finite set.

Choose $\gamma \subset \mathbb{C} - B$
$\uparrow$ simple closed curve

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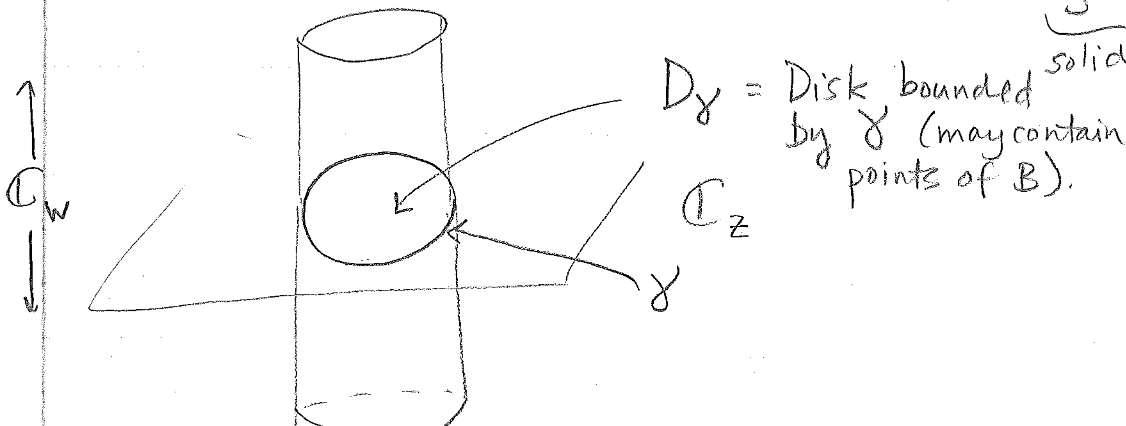
For each $z_\theta \in \gamma$, we have n distinct values of $w \in \mathbb{C}$ satisfying $f(z_\theta, w) = 0$

(γ compact $\Rightarrow \exists\ D_R \subseteq \mathbb{C}$ such that all roots of $f(z_\theta, w)$ for all values of θ are contained in D_R)

$\Longrightarrow$ Get an n-strand braid $\sigma \in \underbrace{\underset{S^1}{\underset{\uparrow}{\gamma}} \times \underset{D^2}{\underset{\uparrow}{D_R}}}_{\text{solid torus.}}$



Also have $\underset{D^2}{\underset{\uparrow}{D_\gamma}} \times \underset{S^1}{\underset{\uparrow}{\partial D_R}}$.

These two solid tori glue along $S^1 \times S^1$ boundary to form an imbedded (PL) copy of $S^3 \subseteq \mathbb{C}^2$ (We shall see that the corresponding closed braid naturally represents a <u>transverse</u> <u>link</u> w/ respect to ξ_{st}, the standard tight contact structure on S^3).

<u>Rudolph, Boileau-Orevkov</u>: Give a topological characterization of which braids can appear via this construction (<u>Quasipositive</u> braids)

It follows from proofs of Milnor conjecture + this work that it is precisely this set of closed braids for which $\tau = \frac{1}{2}s = g_4$.