Goal: Identify a large class of knots for which bound on 4-ball genus coming from $T(K), S(K)$ is SHARP. (Includes Tris for $\eta > 0$)

Lee Rudolph: "Transverse $\mathbb{C}$-links"
(characterization as closures of certain braids).

$\ell_0(z) w^n + \ell_1(z) w^{n-1} + \ldots + \ell_n(z)$

Recall: Let $f(z, w) \in \mathbb{C}[z, w]$ be "sufficiently generic" s.c.c.$\checkmark$ $\gamma \subset \mathbb{C}_z$

Then for each pt. $z_0$ on "generic" s.c.c.$\checkmark$ $\gamma \subset \mathbb{C}_z$

$f(z_0, w) = 0$ has $n$ distinct solutions (assuming for simplicity it misses 0).

Choose $R \gg 0$ so that all solutions for all values of $z_0$ lie in $D_R \subset \mathbb{C}_w$.

\[
\beta_f := \left\{ (z, w) \in \mathbb{C}^2 \mid f(z, w) = 0, z \in \gamma \right\}
\]

forms an $n$-strand braid in $\gamma \times D_R$ over $S^1 \subset D^2$ solid torus.

Remark: $(\gamma \times D_R) \cup (D_R \times \partial D_R) \cup \text{(virtual)} \subset \mathbb{C}_z$

is a genus 1 Heegaard decomposition of $S^3 = \partial (D_\gamma \times D_R)$.

It's a piecewise-smooth imbedding, but by
"smoothing corners" @ $Y \times JDR$, we obtain $\mathbb{C}^2$ as the pullback of complex lines via this smoothed imbedding $S^3 \hookrightarrow \mathbb{C}^2$.

This is precisely the contact structure obtained by applying the Thurston-Winkelnkemper construction to the open book associated to the (fibred) unknot $\subset S^3$.

$z$-axis = $JDR$

$N(U) \approx Y \times D_R$.
$S^3 - N(U) \approx JDR \times D_y$

PW smooth imbedding:
complex lines (2-planes) are
\[ \begin{cases} 
\circ \text{ tangent to } D_R \text{ in } N(U) \\
\circ \text{ tangent to } D_y \text{ in } S^3 - N(U) 
\end{cases} \]

After smoothing: Complex lines are
\[ \begin{cases} 
\circ \text{ tangent to } D_R \text{ along } U \\
\text{(binding)} \\
\circ \text{ tangent to } D_y \text{ along } JDR \\
\text{(one point in each fiber of )} \\
\circ \text{ Everywhere else: Interpolate (twist)}
\end{cases} \]
By construction, $\hat{\beta}_f$ (for $f(z,w) = f_1(z)w^n + \ldots + f_k(z)$) is a transverse representative of $L = \hat{\beta}_f \subset S^3$ with respect to $\mathfrak{Est}$.

Q (Lee Rudolph): Which braids arise as $\beta_f$ for some $f$ as above?

Rudolph's construction: (Turns out to completely answers this question)

Definition: A braid $\beta \in B_n$ is quasipositive if it has a representative word which is a product of arbitrary conjugates of positive elementary Artin generators:

$$\beta = \prod_{j=1}^n \omega_j \sigma_j \omega_j^{-1}$$

where $\omega \in B_n$.

Example: Here's a conjugate of $\sigma_2 \in B_4$ (read left to right, top to bottom: by $\sigma_1 \sigma_2^{-1} \sigma_3$)

Characterization in terms of mapping class represented by $\beta$:

Recall: $B_n = \text{Diff}^+(D_n, \partial D_n, \Delta) / \text{isotopy fixing}$

(pointwise setwise)

(Convention: Since we read braids $L \to R$, braids act ON THE RIGHT)
From this point of view, elementary positive Artin generator \( A_i \)

performs a \( 180^\circ \) CCW rotation of regular neighborhood of arc \( A_i \) connecting points \( p_i, p_{i+1} \).

A conjugate of a positive Artin generator:

performs a \( 180^\circ \) CCW rotation of regular neighborhood of arc \( A_i \).

\[ A = A_i a \]

Note: \( A = A_i a \) for some simple arc \( A_i \).

Claim: Let \( \beta \in \mathbb{B}_n \) be expressed as a product of conjugates of elementary Artin generators (pos. or neg.)

\[ \beta = \prod_{j=1}^{c} \sigma_{i_j}^{\pm} w_j^{-1} \quad (\text{exponents pos.}) \]

Then \( \beta \) bounds a smoothly imbedded surface \( \Sigma \) of Euler characteristic \( \chi \) equal to \( X \) of corresponding “simple” braid (replace each conjugate with elem. generator)

\[ \beta_{\text{simple}} = \prod_{j=1}^{c} \sigma_{i_j}^{\pm} \]

\( X \) of this surface is \( n-c \).
Proof: Movie of the surface (see also "Kamada braid charts" - Scott Carter's talk)

Example from before
\((\sigma_1 \sigma_2^{-1} \sigma_3)(\sigma_2)(\sigma_1 \sigma_2^{-1} \sigma_3)^{-1}\)

one conjugate of a positive generator

Don't change topology of surface

Changes topology of surface.

Call such a surface a BRAINED SURFACE.

Remark: 0. If \(B\) is OP, then it bounds a braided surface with only positive bands.

2. A braided surface is ribbon (i.e., the push-in of an immersed surface in \(S^3\) with only ribbon singularities).
Definition: If \( \hat{\beta} \) bounds a braided surface imbedded in \( S^3 \) (no ribbon singularities), then \( \beta \) is called \( \text{STRONGLY quasipositive} \).

E.g., Torus knots/links, all pos. braid closures.

Rudolph: Called \( \text{QP} \).

Seifert surface \( L = \hat{\beta} \) for \( \beta \) a QP braid \( \iff L = \hat{\beta}_f \) for some \( f(z,w) \in \mathbb{C}[z,w] \) as \( @ \) beginning of lecture.

\[ f(z,w) = f_0(z)w^n + \ldots + f_n(z). \]

Theorem (Rudolph): A quasipositive braided surface can be imbedded as a full subsurface of the (QP) braided surface of a torus knot \( T_r,s \) for suff. large \( r,s \).

(Idea of) Proof: Analyze link \( @ \infty \) to see a "QP braided cobordism from \( \beta \) to \( T_{r,s} \). Let \( r = n, s > r \)."

Remark: For QP Seifert surfaces, this fact can be seen directly without appealing to complex analysis. (cf. Sect. 2.4 of Rudolph's survey in "Handbook of Knot Theory").

Theorem: (Livingston, Plamenovskaya, Shumakovich)

Let \( K \subseteq S^3 \) be QP. Then

\[ \tau(K) = \frac{1}{2} s(K) = g_4(K) \]

\[ \text{HF} \quad \text{Khomur} \quad \text{Rasmussen-Lee} \]

If \( K \) is strongly QP, we also have \( g_4(K) = g(K) \).
Let \( T \) stand for either \( T_1 \) or \( T_2 \).

Proof: Recall that \( T_1, T_2 \) both satisfy

\[
\begin{align*}
|T(K)| &\leq g_4(K) \\
T(K_1 \# K_2) &\leq T(K_1) + T(K_2) \\
T(-K) &\leq -T(K) \\
T(T_{r,s}) &\leq \frac{1}{2}(r-1)(s-1).
\end{align*}
\]

Suppose \( K \) is \( \mathcal{Q}P \). Rudolph's theorem tells us that \( K \) bounds a braided surface \( G \), that is a full subsurface \( F \) of the minimal genus braided surface for \( T_{r,s} \).

\[\Rightarrow\] there is a cobordism \( S^3 \times I \) from \( K \) to \( T_{r,s} \) of genus \( g(F) - g(G) \).

\[\Rightarrow\]

\[
\begin{align*}
T(-K \# T_{r,s}) &\leq g_4(-K \# T_{r,s}) \leq g(F) - g(G) \\
-T(K) + T(T_{r,s}) &\geq T(T_{r,s})
\end{align*}
\]

\[\Rightarrow\]

\[
g(G) \leq T(K)
\]

But we already know that

\[
T(K) \leq g_4(K) \leq g(G).
\]

So \( T(K) = g_4(K) = g(G) \), as desired. \( \blacksquare \)