(p.1)

$\boxed{4/28/14}$

Goal: Identify a large class of knots for which bound on 4-ball genus coming from $\tau(K)$, $s(K)$ is SHARP. (Includes $T_{r,s}$ for $r,s>0$)

Lee Rudolph: "Transverse $\mathbb{C}$-links"
(characterization as closures of certain braids).

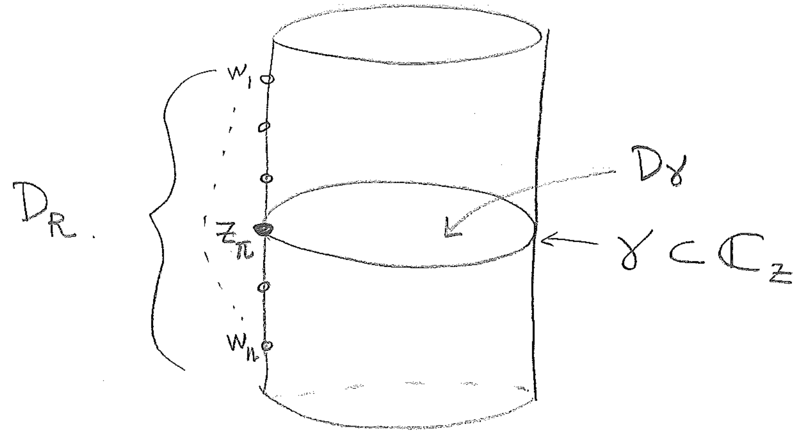
$$f_0(z)w^n + f_1(z)w^{n-1} + \ldots + f_n(z)$$

Recall: Let $f(z,w) \in \mathbb{C}[z,w]$ be "sufficiently generic"

Then for each pt. z_θ on "generic" s.c.c. $\gamma \subset \mathbb{C}_z$ (assume, for simplicity, it misses 0).

$f(z_\theta, w) = 0$ has $\underline{n}$ distinct solutions

Choose $R \gg 0$ so that all solutions for all values of z_θ lie in $D_R \subset \mathbb{C}_w$.



$\beta_f := \{ (z,w) \in \mathbb{C}^2 \mid f(z,w) = 0,\ z \in \gamma \}$ forms an

n-strand braid in $\underbrace{\overset{S^1}{\gamma} \times \overset{D^2}{D_R}}_{\text{solid torus}}$.

Remark: $(\gamma \times D_R) \cup_{\gamma \times \partial D_R} (D_\gamma \times \partial D_R)$

D_γ: disk $\subset \mathbb{C}_z$ containing 0

is a genus $\underline{1}$ Heegaard decomposition of $S^3 = \partial(D_\gamma \times D_R)$.

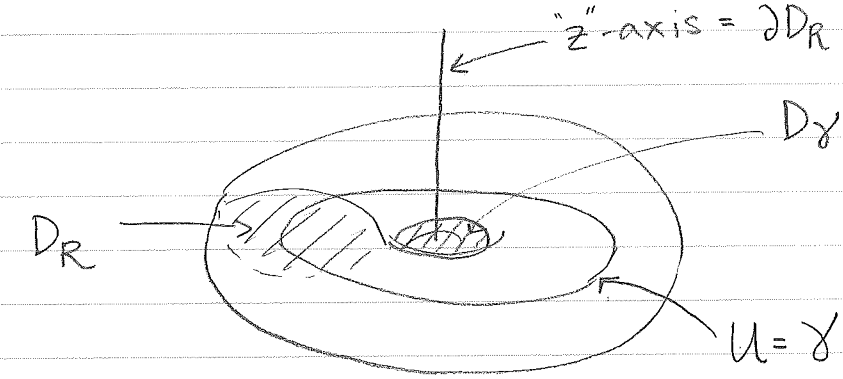
It's a piecewise-smooth imbedding, but by "smoothing corners" @ $\overbrace{\gamma \times \partial D_R}^{\text{Heegaard torus}}$, we obtain

ξ_{st} as the pullback of complex lines via this smoothed imbedding

(isotopy class of) standard tight contact structure on S^3

$$S^3 \hookrightarrow \mathbb{C}^2.$$

This is precisely the (isotopy class of) contact structure obtained by applying the Thurston–Winkelnkemper construction to the open book associated to the (fibred) unknot $\subseteq S^3$:



$N(U) \approx \gamma \times D_R.$

$S^3 - N(U) \approx \partial D_R \times D_\gamma$

PW smooth imbedding: complex lines (2-planes) are
- tangent to D_R in $N(U)$
- tangent to D_γ in $S^3 - N(U)$.

After smoothing: Complex lines are
- tangent to D_R along U (binding)
- tangent to D_γ along ∂D_R (one point in each fiber of OB)
- Everywhere else: Interpolate (twist)

4/28/14

$\Rightarrow$ By construction, $\hat{\beta}_f$ (for $f(z,w) = f_0(z)w^n + \dots + f_n(z)$) is a transverse representative of $L = \hat{\beta}_f \subseteq S^3$ with respect to ξ_{st}.

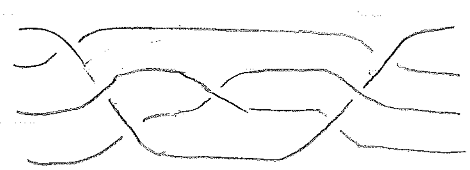
Q (Lee Rudolph): Which braids arise as β_f for some f as above?

Rudolph's construction: (Turns out $\to$ completely answers this question)

Definition: A braid $\beta \in B_n$ is quasipositive if it has a representative word which is a product of arbitrary conjugates of positive elementary Artin generators:

$$\beta = \prod_{j=1}^{c} \omega_j \sigma_{i_j} \omega_j^{-1} \qquad \text{where } \omega \in B_n.$$

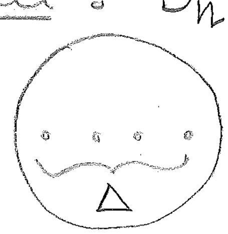
Example: Here's a conjugate of $\sigma_2 \in B_4$ (read left to right, top to bottom: by $\sigma_1 \sigma_2^{-1} \sigma_3$)



Characterization in terms of mapping class represented by β:

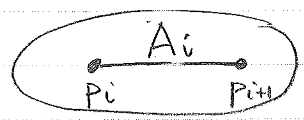
Recall: $B_n = \mathrm{Diff}^+(D_n, \partial D_n, \Delta) /$ isotopy fixing ∂D_n, Δ ptwise.

∂D_n: pointwise; Δ: setwise



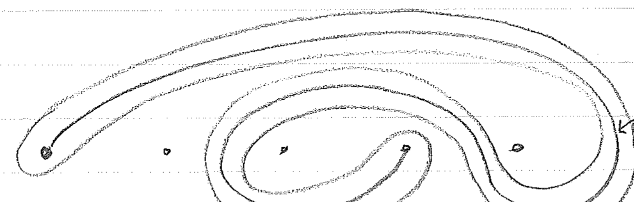
(Convention: Since we read braids $L \to R$, braids act ON THE RIGHT)

From this point of view, elementary <u>positive</u> Artin generator:



performs a $180°$ CCW rotation of regular neighborhood of arc A_i connecting points p_i, p_{i+1}

A <u>conjugate</u> of a positive Artin generator:



performs a $180°$ CCW rotation of reg. nbhd. of <u>arbitrary</u> arc, A.

Note: $A = \underbrace{A_i \omega^{-1}}$ for some simple arc A_i.

<u>Claim</u>: Let $\beta \in B_n$ be expressed as a product of conjugates of elementary Artin generators (pos. or neg.)

$$\beta = \prod_{j=1}^{c} \omega_j \sigma_{i_j}^{\pm} \omega_j^{-1}$$

(QP β has all exponents <u>pos.</u>)

Then $\hat{\beta}$ bounds a smoothly imbedded surface, in B^4, of Euler characteristic (χ) equal to χ of corresponding "simple" braid (replace each conjugate w/ elem. generator).

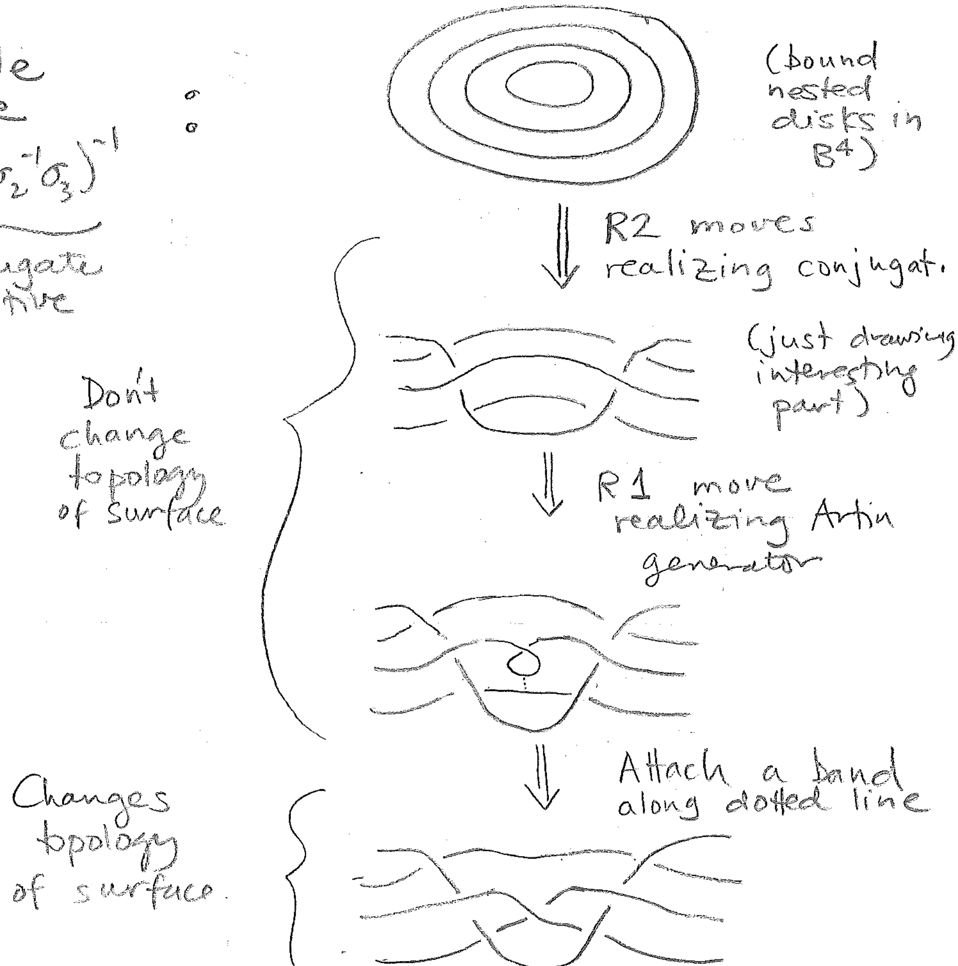
$$\beta_{simple} = \prod_{j=1}^{c} \sigma_{i_j}^{\pm}$$

χ of this surface is $n - c$.

<u>Proof</u>: Movie of the surface (see also "Kamada braid charts" - Scott Carter's talk)

Example from before:

$$\underbrace{(\sigma_1\sigma_2^{-1}\sigma_3)(\sigma_2)(\sigma_1\sigma_2^{-1}\sigma_3)^{-1}}_{\text{one conjugate of a positive generator}}$$

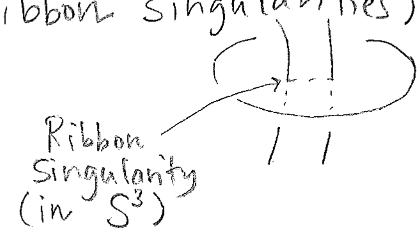


Call such a surface a BRAIDED SURFACE $\square$

<u>Remark</u>: ① If β is QP, then it bounds a braided surface with only positive bands.

Call this a QP <u>braided surface</u>

② A braided surface is <u>ribbon</u> (i.e., the push-in of an immersed surface in S^3 with only ribbon singularities):



Rudolph: Called QP

Definition: If $\hat{\beta}$ bounds a braided surface imbedded in S^3, (no ribbon singularities), then β is called STRONGLY quasipositive (E.g. Torus knots/links, all pos. braid closures)

Seifert surface

Relates QP braids to complex geometry.

$\Longrightarrow$ $\Longleftarrow$

Theorem (Rudolph, Boileau-Orevkov): $L = \hat{\beta}$ for β a QP braid $\Longleftrightarrow$ $L = \hat{\beta}_f$ for some $f(z,w) \in \mathbb{C}[z,w]$ as @ beginning of lecture:

$$f(z,w) = f_0(z)w^n + \dots + f_n(z).$$

Theorem (Rudolph): A quasipositive braided surface can be imbedded as a full subsurface of the (QP) braided surface of a torus knot $T_{r,s}$ for suff. large r,s.

(Idea of) Proof: Analyze link @ ∞ to see a "QP braided cobordism from $\hat{\beta}$ to $T_{n,n}$. Let $r=n$, $s>n$ and rel. prime.

Remark: For QP Seifert surfaces, this fact can be seen directly without appealing to complex analysis. (cf. Sect. 2.4 of Rudolph's survey in "Handbook of Knot Theory") $\square$

Theorem: (Livingston, Plamenevskaya, Shumakovitch) Let $K \subseteq S^3$ be QP. Then

$$\underset{\substack{\uparrow \\ \text{HF}}}{\tau(K)} = \tfrac{1}{2}\underset{\substack{\uparrow \\ \text{Khovanov} \\ \text{Rasmussen-Lee}}}{s(K)} = g_4(K)$$

If K is strongly QP, we also have $g_4(K) = g(K)$.

(from Livingston)

Let "$\mathcal{T}$" stand for either

4/28/14

Proof: Recall that $\overbrace{\tau, \tfrac{1}{2}s}$ both satisfy

- $|\mathcal{T}(K)| \leq g_4(K)$
- $\mathcal{T}(K_1 \# K_2) = \mathcal{T}(K_1) + \mathcal{T}(K_2)$
- $\mathcal{T}(-K) = -\mathcal{T}(K)$.
- $\mathcal{T}(T_{r,s}) = \frac{1}{2}(r-1)(s-1)$.

Suppose K is QP. Rudolph's theorem tells us that K bounds a braided surface, G, that is a full subsurface, F, of the minimal genus braided surface for $T_{r,s}$.

$\Longrightarrow$ there is a cobordism $\subseteq S^3 \times I$ from K to $T_{r,s}$ of genus $g(F) - g(G)$.

$\Longrightarrow$

$$\underset{-\mathcal{T}(K) + \mathcal{T}(T_{r,s})}{\mathcal{T}(-K \# T_{r,s})} \leq g_4(-K \# T_{r,s}) \leq \underbrace{g(F)}_{\mathcal{T}(T_{r,s})} - g(G)$$

$$\Longrightarrow \quad g(G) \leq \mathcal{T}(K)$$

But we already know that

$$\mathcal{T}(K) \leq g_4(K) \leq g(G).$$

So $\quad \mathcal{T}(K) = g_4(K) = g(G)$, as desired. $\square$