p. 1

4/30/14

TODAY: Discuss what Khovanov homology can (so far) tell us about braids and their closures.
B_n = n-strand braid group.

Recall: Diagram $\mathcal{D}(L)$ for $L \subseteq S^3$ $\rightsquigarrow$ $CKh(\mathcal{D}(L))$

Khovanov chain cpx. (over $\mathbb{F} = \mathbb{Q}$) from cube of resolutions
Admits a deformation (Lee).

Plamenevskaya: If $\mathcal{D}(L) = \mathcal{D}(\hat{\sigma})$ for some $\sigma \in B_n$ $\rightsquigarrow$ Distinguished generator of $CKh(\mathcal{D}(\hat{\sigma}))$ which is a cycle $\Rightarrow$ represents an elt. of $Kh(\mathcal{D}(\hat{\sigma}))$ (well-defined up to $\pm$)

This element $\psi(\hat{\sigma}) \in Kh(\mathcal{D}(\hat{\sigma}))/\{\pm 1\}$ is called

Plamenevskaya's invariant.

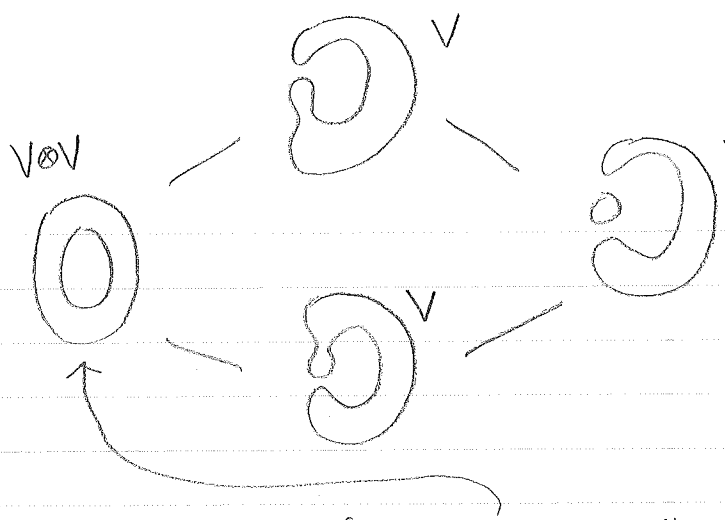
Theorem: (Plamenevskaya) $\psi(\hat{\sigma})$ is an invariant of the transverse isotopy class of $\hat{\sigma} \subseteq (S^3, \xi_{st})$.

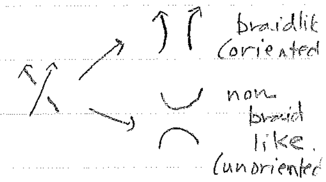
GOALS: Explain how to use $\psi(\hat{\sigma})$ to
1. Prove "slice-Bennequin inequality" (version of adjunction inequality: relates slice genus to $\overline{sl}(L = \hat{\sigma})$). and show that the inequality is sharp for QP links
2. Detect the trivial braid.

Definition of $\psi(\hat{\sigma})$:

Example:  $L = \hat{\sigma}$ $\sigma = \sigma_1^2 \in B_2$.

$V = \mathbb{F}[x]/x^2$





Distinguished generator: $x_{br} := x \otimes \cdots \otimes x$ on "all-braid-like" resolution

Lemma: $\partial^{Kh}(x_{br}) = 0$

Proof: All edges leaving braid-like resolution correspond to merge/multiplication maps, and $m(x \otimes x) = 0$ □.

$\Rightarrow \quad [x_{br}] \in Kh(\hat{\sigma})$

(As usual: Only interesting if this element is an invariant of more than the particular braid closure diagram we chose.)

(Bennequin) Transverse Alexander theorem: Any transverse link (everywhere transverse to ξ_{std}) $T \subseteq (S^3, \xi_{std})$ is transversely isotopic to a closed braid.

(Wrinkle, Orevkov-Shevchishin) Transverse Markov theorem: If $\hat{\sigma}, \hat{\sigma}'$ represent transversely isotopic links, then σ and σ' are related by a finite sequence of braid isotopies, conjugations and positive de/stabilizations.

Theorem: (Plamenevskaya) $\psi(\hat{\sigma}) = [x_{br}] \in Kh(\hat{\sigma})/\pm 1$ is an invariant of the transverse isotopy class of $\hat{\sigma}$.

(Sketch of) Proof: (Invertible) Maps induced by
- R3 moves } in complement — Braid isotopies
- R2 moves } of braid axis — Isotopies and conjugations
- Positive R1 moves over braid axis — Positive de/stabilization

send $[x_{gr}] \xrightarrow{\pm} [x'_{gr}]$

(Note: Cobordism maps in Kh only well-defined up to $\pm$) □

4/30/14

Proposition: (Plamenevskaya) $gr_q(\psi(\hat{\sigma})) = sl(\hat{\sigma})$.

Recall: The self-linking number $sl(T)$ of a transverse link T, bounding a surface Σ, is the linking # of a pushoff, T', of T in the direction of a nonzero vector field $v|_\Sigma \subseteq \xi$ (contact plane)

(Recall: A two-plane field over a 2-mfd. w/ boundary is always trivial).

Lemma: (cf. Etnyre's survey "Legendrian & Transversal Knots") If $T \subseteq (S^3, \xi_{std})$ is a transverse link represented by $\hat{\sigma}$, then $sl(T) = \underbrace{w(\sigma)}_{\text{sum of exponents of Artin generators}} - \underbrace{n(\sigma)}_{\text{braid index}}$

Proof of Proposition: Calculation $\square$

Corollary (Slice-Bennequin inequality: First proved by Rudolph, using Kronheimer-Mrowka):

Let $K \subseteq S^3$ be a knot, and let

$$SL(K) = \max\{ sl(T) \mid T \text{ a transverse rep. of } K \text{ in } \xi_{std} \}.$$

$$SL(K) \leq 2g_4(K) - 1$$

Proof (Plamenevskaya):

Let $\hat{\sigma}$ represent (a transverse isotopy class of) L.
Recall: Lee generator, $s_o \pm s_{\bar{o}}$, in $s_{min}(K)$ is x_{gr} + "lower order" terms in filtration $\Rightarrow$
$sl(\hat{\sigma}) + 1 = gr_q(x_{gr}) + 1 \leq s(K) \leq 2g_4(K)$

So $\quad SL(K) + 1 \leq 2g_4(K)$

$$\underbrace{SL(K)}_{\text{relative Euler number}} \leq \underbrace{2g_4(K) - 1}_{-\chi(\text{slice surface})}.$$

(Think of this as an adjunction inequality). $\square$

I hope this attribution is right. Other names: Rudolph, Livingston, Shumakovitch, Hedden.

Corollary: (Plameneuskaya)
Let $K \subseteq S^3$ be quasipositive. Then
$$SL(K) + 1 = s(K) = 2g_4(K)$$
($\exists$ Similar statement replacing $s(K)$ with $2\tau(K)$).

Proof: If $K = \hat{\sigma}$ for σ QP, then the braided QP surface has $\chi = n - c$
$= -sl(\hat{\sigma})$.

So
$$\underset{c-n}{\overset{\shortparallel}{sl(\hat{\sigma})}} \leq SL(K) \leq s(K) - 1 \leq \underbrace{2g_4(K) - 1}_{\overset{\shortparallel}{c-n}}$$
(since braided QP surface realizes $g_4(K)$).

So all "$\leq$" are "$=$". $\square$

(Remark: Rudolph originally observed that slice-Bennequin bound was sharp for QP knots, i.e.:
$$SL(K) = 2g_4(K) + 1 \text{ for QP } K.$$

Plameneuskaya (et. al.?) observed that this fact can be used to compute $s(K)$, $\tau(K)$ for QP knots).

One last application: Trivial Braid Detection $\boxed{4/30/14}$

<u>Theorem</u>: (Baldwin–G) Let $\sigma \in B_n$.

$$\sigma = \mathbb{1}_n \quad \text{iff} \quad \psi(\hat{\sigma}) \neq 0 \quad \text{and} \quad \psi(\underset{\sim}{m}(\hat{\sigma})) \neq 0$$

mirror (reverse all crossings)

(Remark: Here it's sufficient to work over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$).

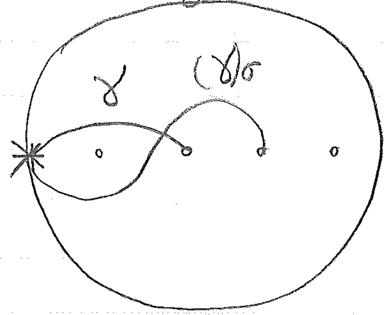
<u>Proof</u>: It's immediate that

$$\sigma = \mathbb{1}_n \implies \psi(\hat{\sigma}) \neq 0, \ \psi(m(\hat{\sigma})) \neq 0.$$

Want to show that $\psi(\hat{\sigma}) \neq 0, \ \psi(m(\hat{\sigma})) \neq 0 \implies \sigma = \mathbb{1}_n$!

Follows from

<u>Key Proposition</u>: If $\psi(\hat{\sigma}) \neq 0$, then σ is "right veering" (RV):



$\forall$ arcs γ from $*$ to Δ, $(\gamma)\sigma$ is "right" of γ

$\hookrightarrow$ When "pulled tight" ($\gamma \pitchfork (\gamma)\sigma$ and no empty extraneous bigons), $(T(\gamma\sigma), T(\gamma))$ agrees w/ disk orientation.

There is an analogous notion of left-veering (LV)

<u>Remarks</u>: (1) σ is $\begin{Bmatrix} RV \\ LV \end{Bmatrix}$ iff $m(\sigma)$ is $\begin{Bmatrix} LV \\ RV \end{Bmatrix}$.

(2) σ is both RV & LV $\iff \sigma = \mathbb{1}$.
(since in this case $\gamma\sigma \sim \gamma \ \forall$ arcs γ).

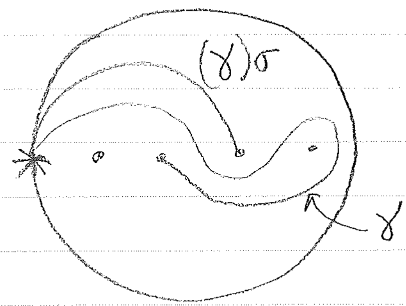
Proof of Theorem (from Proposition):

$$\Psi(\hat{\sigma}), \Psi(m(\hat{\sigma})) \neq 0 \implies \sigma \text{ is RV \& LV}$$
$$\implies \sigma = \mathbb{1}_n. \quad \square$$

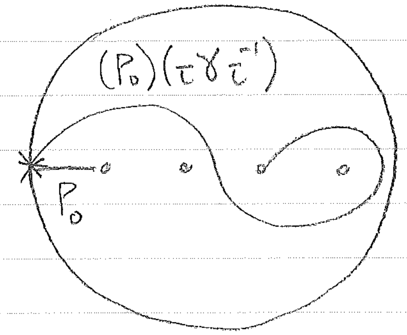
Proof of Proposition: Will prove that if

σ is not RV then $\Psi(\hat{\sigma}) = 0$.

σ not RV $\implies \exists$ arc γ such that $(\gamma)\sigma$ is left of γ



$\implies$ Some conjugate of σ, say $\tau\sigma\tau^{-1}$, sends P_0 to left:



(Fenn, Greene, Rolfsen, Rourke, Wiest) $\implies$

Related to a strict, total order on braid group.

$\left\{\begin{array}{l} \exists \text{ a word representing } \tau\gamma\tau^{-1} \text{ w/} \\ \circ \text{ at least one appearance of } \sigma_1^{-1} \\ \circ \text{ no appearances of } \sigma_1^{+} \end{array}\right.$

$\boxed{\text{Explicit procedure}}$

Enough to produce an explicit chain $\theta \in CKh(\tau\sigma\tau^{-1})$ with

$$\partial\theta = x_{br} \implies [x_{br}] = \psi(\widehat{\tau\sigma\tau^{-1}}) = \psi(\hat{\sigma}) = 0.$$

Conclusion: Gives a new solution to the WORD problem in B_n.

$$B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & |i-j| \geq 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & |i-j| = 1 \end{array} \right\rangle.$$

Given a word w, compute $\psi(\hat{w})$, $\psi(\widehat{m(w)})$.
(just 2 homology classes, so could be reasonably efficient computationally).