Exercises (Last updated April 23, 2014):

Do these if it helps you. Talk to each other. Talk to me.
If you write something down and turn it in, I will read it.

Week one: 1/13-1/17

(1) Find an explicit handlebody decomposition of $\mathbb{C}P^2$, in terms of its description as the set of complex lines in $\mathbb{C}^3$:

$$\mathbb{C}P^2 := \{[x : y : z] \,|\, (x, y, z) \in \mathbb{C}^3 \setminus \{(0, 0, 0\},$$

where $[x : y : z] \sim [hx : hy : hz]$ for $h \in \mathbb{C} \setminus \{0\}$. If you get stuck, see Example 4.2.4 in Gompf-Stipsicz and the discussion after Ex. 4.2.5.

(2) Let K be any nontrivial knot in S^3 and let μ_K denote (the isotopy class of) any *meridian* of K (i.e., the boundary of a fiber of a tubular neighborhood of K).

(a) Show that the Kirby diagram corresponding to marking both K and μ_K by 0's represents the same 4–manifold as the Kirby diagram corresponding to marking both link components of the Hopf link, $H = U \cup \mu_U$ (here U represents the unknot), with 0's.

(b) What 3–manifold is obtained by doing 0–framed Dehn surgery on $L = K \cup \mu_K$?

(c) What 3–manifold is obtained by doing 0–framed Dehn surgery on $H = U \cup \mu_U$?

(3) Let X^4 be a compact, connected, oriented 4–manifold, and let $Y^3 = \partial X^4$ be its closed, connected, oriented boundary. Using the fact (proved in lecture) that every nullhomologous knot $K \subset Y^3$ bounds a smoothly-imbedded orientable surface (*Seifert surface*), prove that K bounds a properly imbedded orientable surface in X^4. (We already have a smooth imbedding of an orientable surface into $\partial X^4 = Y^3$. Show that the imbedding can be made proper via a smooth isotopy.)

(4) Give an alternative direct argument for (3) that a nullhomologous knot $K \subset (Y^3 = \partial X^4)$ bounds a properly imbedded orientable surface in X^4. Explicitly, prove more generally that if M^n is a smooth, compact, connected, orientable n–manifold, then any homology class in $H_{n-2}(M^n, \partial M^n)$ can be represented by a properly imbedded orientable submanifold of dimension $n-2$. Note that any Eilenberg-MacLane space $K(\mathbb{Z}, 2)$ is homotopy-equivalent to $\mathbb{C}P^\infty$, and any continuous map between CW complexes can be approximated by a cellular map (cf. Hatcher's *Algebraic Topology*, Sec. 4.1 and 4.3).

(5) Fix $p, q \in \mathbb{Z}^+$, and consider the complex polynomial, $f(z_1, z_2) = z_1^p + z_2^q$. Let

$$V_f := \{\mathbf{z} \in \mathbb{C}^2 \,|\, f(\mathbf{z}) = 0\}$$

denote the zero set of f, and consider the isolated singularity at $\mathbf{z}_0 = (0, 0)$. Let S^3_ϵ denote the boundary of an ϵ–neighborhood of $(0, 0)$ for sufficiently small ϵ. Show that the associated *link of the singularity* at $\mathbf{z}_0$, $L_f := V_f \cap S^3_\epsilon$, is a (p, q) torus link in S^3_ϵ.

(6) Fix $p, q \in \mathbb{Z}^+$, and realize the (p, q) torus link, $T_{p,q}$, as the closure of the q–strand braid $(\sigma_1 \cdots \sigma_{q-1})^p$. Show that Seifert's algorithm applied to the resulting diagram yields a surface, $F_{p,q}$, with Euler characteristic $p+q-pq$. Conclude that if $\gcd(p, q) = 1$ (so $T_{p,q}$ is a *knot*), $F_{p,q}$ has genus $\frac{1}{2}(p-1)(q-1)$.

Week two: 1/22-1/24

(1) Compute the Khovanov homology (over $\mathbb{Q}$) of the Hopf link (with the two natural choices of orientation on its components).
(2) Compute the Khovanov homology (over $\mathbb{Q}$) of the 2–component unlink using:
  (a) a standard, crossingless diagram
  (b) a 2–crossing diagram (e.g., change one of the crossings in a standard diagram of the Hopf link)

Week three: 2/24-2/28

(1) Show that if L is an ℓ–component link, then its Lee complex is supported in $\mathbb{Z}/4\mathbb{Z}$–valued q–gradings ℓ and $\ell + 2$. (Hint: Verify this statement in the case that L is the ℓ–component unlink, and use the fact that a diagram for any ℓ–component link can be changed to a diagram for the ℓ–component unlink by changing some of its crossings.)
(2) Let $\mathcal{D}$ be a diagram for a knot $K \subset S^3$. Choose any crossing "c" in the diagram, and let $\mathcal{D}(*0)$ (resp., $\mathcal{D}(*1)$) be the diagram obtained from $\mathcal{D}$ by replacing "c" with its 0 (resp., 1) resolution. Verify that one of $\mathcal{D}(*0)$, $\mathcal{D}(*1)$ is a diagram for a knot and the other is a diagram for a 2–component link. Prove that, moreover, the *oriented* resolution is the one representing the 2–component link. (Hint: How are the Seifert surfaces coming from Seifert's algorithm related?)
(3) Suppose that

$$\cdots \xrightarrow{\delta^{i-1}} A^i \longrightarrow B^i \longrightarrow C^i \xrightarrow{\delta^i} \cdots$$

is a bounded long exact sequence of finite-dimensional vector spaces with connecting homomorphisms $\delta^i : C^i \to A^{i+1}$. Let
  - $\dim(A)$ denote $\sum_i \dim(A^i)$ (similarly for $\dim(B)$, $\dim(C)$) and
  - $\mathrm{rk}(\delta)$ denote $\sum_i \mathrm{rk}(\delta^i)$.

Prove that $\dim(B) = \dim(A) + \dim(C) - 2 \cdot \mathrm{rk}(\delta)$.

Week four: 3/10-3/12

(1) We discussed in class on Wednesday, 3/12, how to use relative Morse theory to associate to a knot K in a closed, connected, oriented 3–manifold Y a doubly-pointed Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ compatible with the pair, (Y, K). Show conversely that a doubly-pointed Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ uniquely specifies an (oriented) knot, up to isotopy, as follows.

Let γ_a (resp., γ_b) be an imbedded arc in $\Sigma - \alpha_1 - \ldots - \alpha_g$ (resp., in $\Sigma - \beta_1 - \ldots - \beta_g$) connecting z to w (resp., connecting w to z). Let $\widetilde{\gamma}_a$ (resp., $\widetilde{\gamma}_b$) be the result of pushing γ_a down (resp., up) slightly into the α (resp., β) handlebody. Then $K = \widetilde{\gamma}_a \cup \widetilde{\gamma}_b$. Note that $\gamma_a \cup \gamma_b$ determines a diagram of K on Σ (by asserting that at every crossing of γ_a and γ_b, γ_b is the over-crossing strand). (*You will need to check that there exist arcs $\gamma_a \subset \Sigma - \alpha_1 - \ldots - \alpha_g$ and $\gamma_b \subset \Sigma - \beta_1 \ldots - \beta_g$ connecting*

w and z and that the isotopy class of the resulting knot in Y is independent of the choice of γ_a and γ_b.)

(2) Choose a few favorite non-trivial knots (or, rather, knot diagrams), and construct their associated marked/doubly-pointed Heegaard diagrams as described in class on Wed., 3/12. This construction is also described nicely in Ozsváth-Szabó's *On Heegaard diagrams and holomorphic disks* and Manolescu's survey, *An introduction to knot Floer homology*. See also Ozsváth-Szabó's paper on the arXiv with "Alternating knots" in the title. Draw a diagram of K on Σ as described in the previous problem.

(3) Choose a few favorite non-trivial knots (or, rather, knot diagrams) in n–bridge position and construct their associated genus 0, $2n$-pointed Heegaard diagrams (with $n-1$ α curves and $n-1$ β curves) as discussed in class on Wed., 3/12. Explain how to obtain a doubly-pointed genus $n-1$ Heegaard diagram for K by deleting neighborhoods of $n-1$ of the (w, z) basepoint pairs, identifying the boundaries of the deleted neighborhoods pairwise, and making appropriate alterations to the collection of α, β curves. Explain why your construction does, indeed, yield a doubly-pointed Heegaard diagram for K. If you get stuck, consult the examples in Manolescu's survey and Rasmussen's paper on the arXiv entitled "Floer homology of surgeries on 2–bridge knots."

Weeks five and six: 3/17-3/19, 3/24-3/26

(1) Choose one of the doubly-pointed Heegaard diagrams coming from a marked Heegaard diagram for a knot in S^3 that you constructed last week and use it to write down explicit presentations for $\pi_1(S^3 - N(K))$ (choose any basepoint for π_1 you like here) and its abelianization, $H_1(S^3 - N(K))$, with
   - a single generator for each α curve,
   - a single relation for each β curve except $\beta_g = \mu$, the meridian

(2) Use the marked Heegaard diagram from the previous problem to produce a Heegaard diagram for n–surgery on K for each choice of $n \in \mathbb{Z}$. (*How do you identify which projection of K on Σ is trivial in $H_1(S^3 - N(K))$, i.e., represents the canonical Seifert framing for K?*)

(3) Compute the Alexander gradings of the generators of the knot Floer complex associated to the doubly-pointed Heegaard diagram you used in problem (1).

(4) Consider the following $\mathbb{Z}$–filtered chain complex $\mathcal{C}$ (over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$). The underlying $\mathbb{F}$–vector space has a basis,
$$\mathcal{G} = \{a_1, a_2, b_1, b_2, b_3, c_1, c_2, c_3\},$$
endowed with an $\mathbf{A}$–grading,
$$\mathbf{A} : \mathcal{G} \to \mathbb{Z},$$
given by $\mathbf{A}(a_i) = -1$, $\mathbf{A}(b_i) = 0$, $\mathbf{A}(c_i) = 1$. The boundary map is given by
   - $\partial(a_1) = \partial(a_2) = 0$
   - $\partial(b_1) = a_1$ and $\partial(b_2) = a_1 + a_2 + b_3$ and $\partial(b_3) = 0$
   - $\partial(c_1) = c_2 + c_3$ and $\partial(c_2) = b_3$ and $\partial(c_3) = b_3$

   As usual, the above $\mathbf{A}$–grading endows $\mathcal{C}$ with the structure of a (bounded) $\mathbb{Z}$–filtered complex
$$0 = \mathcal{F}_{-2} \subseteq \mathcal{F}_{-1} \subseteq \mathcal{F}_0 \subseteq \mathcal{F}_1 = \mathcal{C}$$
by defining
$$\mathcal{F}_m := \mathrm{Span}_{\mathbb{F}}\{x \in \mathcal{G} \mid \mathbf{A}(x) \leq m\}$$

with restricted differential. Check that I haven't made any mistakes and that the above does, indeed, define a filtered chain complex. Then compute its homology, the homology of its associated graded complex, $\tau_{min}(\mathcal{C})$, and $\tau_{max}(\mathcal{C})$ (as defined in class on 3/26).

Weeks seven and eight: 3/31-4/2, 4/9-4/11

(1) Choose a few 2–bridge knots, at least one of which is neither $K_{3,1}$ (the trefoil) nor $K_{5,2}$ (the figure-8), and compute their CFK^∞ complexes from the genus 1 doubly-pointed Heegaard diagrams we discussed in class on 3/17. For each, use the Ozsváth-Szabó/Rasmussen "Large surgery" theorem to compute $\widehat{HF}(S^3_{\pm p};\mathfrak{s}_m)$ for sufficiently large $p \in \mathbb{Z}^+$. Among the knots you studied, which have (sufficiently large integral) surgeries which are L–spaces (rank of $\widehat{HF}$ is 1 in each Spin^c structure)?

(2) Suppose we have a 2-pointed Heegaard diagram $(\Sigma, \vec{\alpha}, \vec{\beta}, w, z)$ for (Y, K). We discussed in class on 4/11 how to do a simultaneous (0-1), (2-3) birth (i.e., a simultaneous birth of a pair of canceling 0 & 1 and 2 & 3 handles) to produce an extra pair of intersections of K with Σ, yielding a 4–pointed Heegaard diagram for (Y, K) (more generally, the construction we describe works to produce a $2(n + 1)$–pointed Heegaard diagram for K from a $2n$–pointed Heegaard diagram for K).

Explicitly, referring to the notation in Problem (1) from Week four, choose any pair of adjacent points along γ_a (where "adjacent" means there are no intersection points with β curves in-between) and name them z', w'. Let α' be the boundary of a regular neighborhood of the subset of γ_a connecting w' to z and β' be the boundary of a regular neighborhood of the subset of γ_a connecting z' to w'. Then the desired 4–pointed Heegaard diagram is $(\Sigma, \vec{\alpha} \cup \alpha', \vec{\beta} \cup \beta', (w, w'), (z, z'))$.

(Remark: one can also use adjacent points along γ_b, by doing a search/replace $\gamma_a \leftrightarrow \gamma_b$, $\beta \leftrightarrow \alpha$, $\beta' \leftrightarrow \alpha'$, $z \leftrightarrow w$, $z' \leftrightarrow w'$ in the description above.)

Show that we can always replace (0-1), (2-3) de/stabilizations as above by (1-2) de/stabilizations in the following strong sense. Any sequence involving

- a certain number of (0-1),(2-3) stabilizations
- handleslides/isotopies among α, β curves
- the same number of (0-1),(2-3) destabilizations

can be replaced by a sequence of the form

- a certain number of (1-2) stabilizations
- handleslides/isotopies among α, β curves
- the same number of (1-2) destabilizations

(i.e., the resulting Heegaard diagrams after the two sequences will coincide).

(3) Let $\mathcal{C}$ be a $\mathbb{Z}^2$–filtered chain complex, finitely and freely generated over $\mathbb{F}[U, U^{-1}]$, where the $\mathbb{Z}^2$ filtration is induced in the standard way by a $\mathbb{Z}^2$ grading

$$\mathbf{A} : \mathcal{G} \to \mathbb{Z}^2$$

on the finite set, $\mathcal{G}$, of generators, and $\mathbf{A}(U\mathbf{x}) = \mathbf{A}(\mathbf{x}) - (1, 1)$ for all $\mathbf{x} \in \mathcal{G}$. Prove that if $\mathcal{C}'$ is $\mathbb{Z}^2$–filtered chain homotopy equivalent to $\mathcal{C}$, then $\mathcal{C}\{i = 0\}$ and $\mathcal{C}'\{i = 0\}$ are $\mathbb{Z}$–filtered chain homotopy equivalent. (Corollary: Knowing that the $\mathbb{Z}^2$–filtered chain homotopy type of $CFK^\infty(Y, K)$ is an invariant of (Y, K) implies that the $\mathbb{Z}$–filtered chain homotopy type of $\widehat{CF}(Y, K)$ is an invariant of (Y, K).

(4) Let $\mathcal{C}$ be any $\mathbb{Z}^2$–filtered chain complex arising as $CFK^\infty(K)$ for $K \subset S^3$, and let $\mathcal{C}^*$ be the complex defined by reversing all arrows and reflecting across the slope -1 line through the origin. Prove that $\tau(\mathcal{C}^*) = -\tau(\mathcal{C})$. (Corollary: If K is a knot in S^3 and $-K$ denotes the orientation reverse of its mirror, then $\tau(-K) = -\tau(K)$).

(5) Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be as above. Prove that $\tau(\mathcal{C}_1 \otimes \mathcal{C}_2) = \tau(\mathcal{C}_1) + \tau(\mathcal{C}_2)$. (Corollary: If K_1, K_2 are knots in S^3, then $\tau(K_1 \# K_2) = \tau(K_1) + \tau(K_2)$.)