Exercises (Last updated March 21, 2016):
Do these if it helps you. Talk to each other. Talk to me.
If you write something down and turn it in, I will read it.

Week one: 1/18-1/22
(1) (a) Let $K \subseteq S^3$ be a knot. Prove that $H^1(S^3 - N(K)) \cong \mathbb{Z}$.
(b) Verify that the analogue of the above is true when $S^3$ is replaced by any $\mathbb{Z}$–homology sphere (i.e., a closed 3–manifold $Y$ with $H_*(Y) \cong H_*(S^3)$).
(c) What can you say in the case when $K$ is a nullhomologous knot in a $\mathbb{Q}$–homology sphere $Y$ (replace the homology condition with $H_*(Y; \mathbb{Q}) \cong H_*(S^3; \mathbb{Q})$)? A non-nullhomologous knot? What if $Y$ is not a $\mathbb{Q}$–homology sphere?

(2) Fix $p,q \in \mathbb{Z}^+$, and consider the complex polynomial, $f(z_1, z_2) = z_1^p + z_2^q$. Let $V_f := \{z \in \mathbb{C}^2 | f(z) = 0\}$ denote the zero set of $f$, and consider the isolated singularity at $z_0 = (0,0)$. Let $S^3_\epsilon$ denote the boundary of an $\epsilon$–neighborhood of $(0,0)$ for sufficiently small $\epsilon$. Show that the associated link of the singularity at $z_0$, $L_f := V_f \cap S^3_\epsilon$, is a $(p,q)$ torus link in $S^3$.

(3) Fix $p,q \in \mathbb{Z}^+$, and realize the $(p,q)$ torus link, $T_{p,q}$, as the closure of the $q$–strand braid $(\sigma_1 \cdots \sigma_{q-1})^p$. Show that Seifert’s algorithm applied to the resulting diagram yields a surface, $F_{p,q}$, with Euler characteristic $p + q - pq$. Show that if gcd$(p,q) = 1$, then $T_{p,q}$ is a knot, and $F_{p,q}$ has genus $\frac{1}{2}(p-1)(q-1)$.

Weeks two and three: 1/25-2/5
(1) Realize the RH trefoil knot (aka $T_{2,3}$) as the closure of the (oriented) index 2 braid $\sigma_1^7$, and use the process described in class on 1/27 to show that $T_{2,3}\# - T_{2,3}$ is not only slice but ribbon. Visualize the corresponding ribbon-immersed surface in $S^3$.

(2) Draw a ”movie” of an obvious ribbon disk bounded by $K$, the stevedore’s knot (use the diagram drawn in class on 1/29). That is, let $D \subset B^4$ be a smoothly, properly-embedded disk bounded by $K$ such that the radial function $f : B^4 \to [0,1]$ restricts to a Morse function on $D$ with no critical points of index 2. Choose sufficiently many $t_1 < t_2 < \ldots < t_k \in [0,1]$ such that the diagrams $f^{-1}(t_1), \ldots , f^{-1}(t_k)$ give a reasonable ”movie” of the disk via its slices. Convince yourself (if you are not already convinced) that this disk is imbedded in $B^4$.

(3) Suppose $K \subset S^3$ is a knot, and $(D, \partial D) \subset (V, S^3)$ is a properly-embedded disk in $V$, an integer homology $B^4$ (that is: $H_*(V) \cong H_*(B^4)$).
(a) Explain why $H_1(V - N(D)) \cong \mathbb{Z}$, generated by a meridian of $K$.
(b) Explain why $\partial (V - N(D))$ is 0–surgery on $K$.

Week four: 2/10-2/12
(1) Compute the Khovanov homology (over $\mathbb{C}$) of the Hopf link (with the two choices of orientation on its components).

(2) Compute the Khovanov homology (over $\mathbb{C}$) of the 2–component unlink using:
(a) a standard, crossingless diagram
(b) a 2-crossing diagram (e.g., change one of the crossings in a standard diagram of the Hopf link)

**Weeks five and six: 2/17-3/4**

(1) Let \((C, \partial)\) be a finite-dimensional \(\mathbb{Z}\)-filtered chain complex over a field \(\mathbb{F}\) whose filtration is induced in the standard way by a \(\mathbb{Z}\) grading

\[
\text{gr} : G \to \mathbb{Z}
\]

on a distinguished basis for \(C\) (as described in class on 3/2). In particular, the differential \(\partial\) can be decomposed according to degree:

\[
\partial = \sum_{i=0}^{N} \partial_i,
\]

where \(\text{deg}(\partial_i) = i\). Let \((C_w, \partial_w)\) be the chain complex (over \(\mathbb{F}[w]\)) defined by:

- \(C_w := C \otimes_{\mathbb{F}} \mathbb{F}[w]\)
- \(\partial_w := \sum_{i=0}^{N} w^i \partial_i\)
- \(\text{deg}(w) = -1\)

(a) Verify that \((C_w, \partial_w)\) is a graded chain complex (i.e., \(\partial_w^2 = 0\), and \(\text{deg}(\partial_w) = 0\)), and with respect to the distinguished basis the boundary maps have monomial entries.

(b) Let \(\theta \in C\) be a cycle satisfying \([\theta] \neq 0 \in H_0(C)\). Show that there exists a cycle \(\theta_w \in C_w\) satisfying \(w^k[\theta_w] \neq 0\) for all \(k \in \mathbb{Z^{\geq 0}}\).

(c) Recall that \(\text{gr}(\theta) := \max\{n \in \mathbb{Z} \mid \exists \theta' \in F_n \text{ such that } [\theta] = [\theta']\}\). Show that

\[
\text{gr}(\theta) = \max\{\text{gr}(\theta_w') \mid w^k[\theta_w'] = [\theta_w] \text{ for some } k \in \mathbb{Z^{\geq 0}}\}\}