PROPERTIES OF THE HANOI GRAPH FOR 4 PEGS

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Abstract. For the Tower of Hanoi problem, it has been found that it is possible to construct a nice graph with all possible states as the vertices and all legal moves between possible states as the edges between corresponding vertices. This "Hanoi graph" we will call H_n^k for k pegs and n disks. Many properties of this graph are well-known for k=3, but not for $k \geq$4.

1. Background Information

In this article, we will follow the convention of [POOLE] to label vertices. Each vertex corresponds to a configuration of disks on the pegs. The pegs are labeled $0, 1, ..., k-1$ and the disks are labeled from smallest to largest $0, 1, ..., n-1$. A configuration of disks corresponds to an n-bit string $a_{n-1}...a_1a_0$ where $a_i \in \{0, 1, ..., k-1\}$ and $a_i = j$ if disk i lies on peg j.

Definition 1.1. A *perfect state* is a configuration of disks in which all disks are on one peg. There are k perfect states, 00...0, 11...1, etc. which can also be denoted **0**, **1**, ..., **k-1**.

Definition 1.2. $[x]$ is a *block*, or the set of all vertices whose labels begin with x. This corresponds to all states in which the largest disk is on peg x.

Each block is a subgraph of H_n^k isomorphic to H_{n-1}^k, thus, H_n^k is made up of k copies of H_{n-1}^k joined by edges. Similarly, since each block is comprised of k sub-blocks, H_n^k is made up of k^j copies of H_{n-j}^k joined by edges.

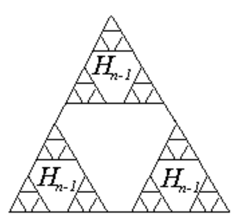


Figure 1. H_n^3 is a fractal-like triangle

Definition 1.3. An edge is called a *bridge* if one end vertex is in $[x]$ and the other end vertex is in $[y]$ where $x \neq y$.

The recursive structure of H_n^k for $k > 3$ offers some additional complexity, as there are more bridges as k increases.

Date: July 31, 2008.

Definition 1.4. $B_{x,y}$ is the set of all vertices which correspond to states in which disks are only on pegs x and y.

It is easy to see that $B_{x,y}$ contains 2^n vertices. Also, $zB_{x,y}$ will be defined as the set of all vertices which correspond to states in which all disks are on pegs x and y except for the largest disk, which is on peg z. For example, $201011 \in 2B_{0,1}$.

Theorem 1.5. *The total number of bridges for H_n^k is exactly $\binom{k}{2}(k-2)^{n-1}$.*

For $k = 3$ this is 3 and for $k = 4$ this is $6 \times 2^{n-1}$.

Proof. There are $\binom{k}{2}$ ways to choose two from k blocks. There exist bridges between two blocks, say [0] and [1], if and only if they are edges between the vertices described by $0B_{2,3,\ldots,k-1}$ and $1B_{2,3,\ldots,k-1}$. There are 2^{n-1} vertices each in $0B_{2,3,\ldots,k-1}$ and $1B_{2,3,\ldots,k-1}$, and each vertex has exactly one bridge (to one vertex in the other block). Thus, given two blocks, there are exactly $(k-2)^{n-1}$ bridges between them. □

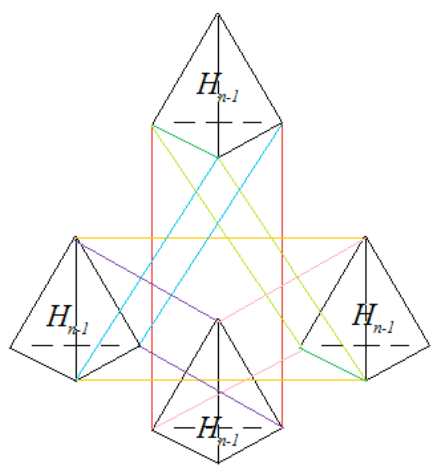


FIGURE 2. the bridges are all the colored edges. We can see the bridges between [0] and [1] as the light-green edges. We can also see that they only exist between $0B_{2,3}$ and $1B_{2,3}$, the green bottom-left sides of [0] and [1], respectively

2. Exploring the Properties of the H_n^4 Graph

Let $deg(v)$ denote the degree of vertex v, $\Delta(G)$ denote the maximum degree of all the vertices of graph G, and $\delta(G)$ denote the minimum degree of all the vertices of graph G. Thus, $\delta(H_n^3) = deg(a) = 2$ where a is any perfect state in H_n^3. $\Delta(H_n^3) = deg(b) = 3$ where b is any non-perfect state in H_n^3.

Conjecture 2.1. *For all n, $\delta(H_n^4) = deg(a) = 3$ where a is a perfect state in H_n^4. For $n \geq 3$, $\Delta(H_n^4) = deg(b) = 6$ where $b \in x_0B_{x_1,x_2}$ and $x_0 \neq x_1 \neq x_2$.*

Furthermore, for all n, $\delta(H_n^k) = deg(c) = k-1$ where c is any perfect state in H_n^k. For $n \geq k-1$, $\Delta(H_n^k) = deg(d) = \sum_{i=1}^{k-1} i$ where $d \in x_0 B_{x_1,x_2,\ldots,x_{k-2}}$.

Let the *distance* between a and b, denoted by $d(a,b)$, be defined as the number of edges in a shortest path between vertex a and vertex b.

Conjecture 2.2. *In H_n^4, given $a \in B_{1,2}$, $d(a, \mathbf{0})$ is maximized when $a = \mathbf{1}$ or $\mathbf{2}$.*

Remark 2.3. In H_n^4, there exist two vertices $a \in [0]$, $b \in [1]$, in which the shortest path between them passes through [2] or [3]. This means that the largest disk moves twice in a geodesic.

For example, $a = 0111$ and $b = 1000$.

Open Question 2.4. *In H_n^4, there exist two vertices $a \in [0]$, $b \in [1]$, in which the shortest path between them passes through [2] and [3]. This means that the largest disk moves three times in a geodesic.*

3. Graphing the Frame-Stewart Solution

The Frame-Stewart algorithm is theorized to give the optimal solution from one perfect state to another for 4 pegs. The Frame-Stewart solution from perfect state $\mathbf{0}$ to perfect state $\mathbf{1}$ is given as follows.

(1) Using all pegs, move the smallest $n - l_n$ disks from peg 0 to peg 2.
(2) Using the remaining pegs 0, 1 and 3, move the largest l_n disks from peg 0 to peg 1.
(3) Using all pegs, move the smallest $n - l_n$ disks from peg 2 to peg 1.

A table of values for l_n, $n - l_n$ for low values of n is given below. l_n is the smallest integer such that $n \leq \frac{l_n(l_n+1)}{2}$ (cf. [MR]). An unadorned arrow denotes a series of moves from one state to another using all pegs and an arrow with superscripts x, y and z denotes a series of moves using only pegs x, y and z.

n	l_n	$n - l_n$	sequence of moves
1	1	0	$0 \longrightarrow 1$
2	2	0	$00 \xrightarrow{0,1,3} 11$
3	2	1	$000 \longrightarrow 002 \xrightarrow{0,1,3} 112 \longrightarrow 111$
4	3	1	$0000 \longrightarrow 0002 \xrightarrow{0,1,3} 1112 \longrightarrow 1111$
5	3	2	$00000 \longrightarrow 00022 \xrightarrow{0,1,3} 11122 \longrightarrow 11111$
6	3	3	$000000 \longrightarrow 000222 \xrightarrow{0,1,3} 111222 \longrightarrow 111111$

We will introduce a new class of graphs in which H_n^k is partitioned into "layers".

Definition 3.1. When there exists a restriction on H_n^k in which a peg m is *fixed*, disks may not be moved to or from peg m.

For example, fixing peg 2, an edge does not exist between 00 and 02, but an edge exists between 02 and 12. Layers are subgraphs of H_n^4 that are visible when a peg m is fixed.

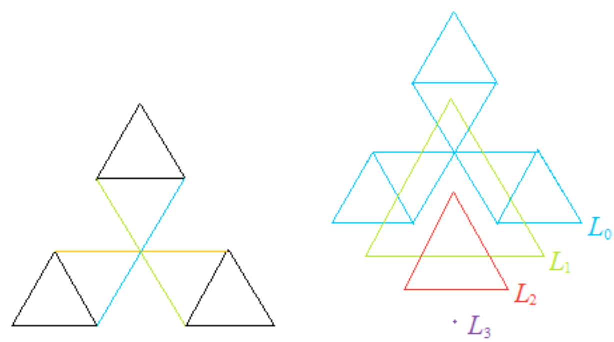
Definition 3.2. The vertex set of a *layer* is a set of vertices in H_n^k in which every vertex in the set can be reached from another vertex in the set through a move that does not use peg m. The edge set of a *layer* is the set of all edges between the vertices in the aforementioned vertex set.

Finally, we define the Partitioned Hanoi graph $PH_n^k(m)$.

Definition 3.3. $PH_n^k(m)$ is the (pairwise disjoint) union of all layers in H_n^k when peg m is fixed.

Lemma 3.4. *Each layer of $PH_n^4(m)$ is isomorphic to H^3.*

Proof. Since there are a total of 4 pegs and peg m is fixed, there are 3 non-fixed pegs. If i disks are on peg m, since disks cannot be moved onto or off peg m, there will always be i disks on peg m in the layer. Thus, all possible moves in this layer will only involve $n-i$ disks on 3 pegs, so the layer is isomorphic to H_{n-i}^3. □



(a) $B_{0,1,3}$ for $n=2$ (b) layers in H_2^4

FIGURE 3. $B_{0,1,3}$ and $PH_2^4(2)$

As we can see in figure 3(b) these layers of H^3 subgraphs are embedded in the H_n^4 graph in figure 2. Each layer is a "twisted" non-planar version of the H_n^3 graph in figure 1 but they are otherwise identical. To label these layers, we will find a set of vertices with the property that there is a unique vertex from this set in each layer.

Lemma 3.5. *There is a unique vertex from $B_{x,m}$ where $x \neq m$ in each layer in $PH_n^k(m)$.*

Vertices in $B_{x,m}$ where $x \neq m$ can be ordered in a binary-like progression:

$$xx...xxx,\ xx...xxm,\ xx...xmx,\ xx...xmm,\ \ldots,\ mm...mmm$$

Proof. Every vertex of $B_{x,m}$ is in $PH_n^k(m)$, so each vertex of $B_{x,m}$ is in a layer. If peg m is fixed, it is impossible for there to be edges between any two vertices in $\{xx...xxx,\ xx...xxm,\ xx...xmx,\ xx...xmm,\ \ldots,\ mm...mmm\}$. Thus, there cannot be two vertices from $B_{x,m}$ in the same layer. □

Convention 3.6. We convert each of these binary-ordered vertices in $B_{x,m}$ where $x \neq m$ into a binary number as follows: x becomes 0, m becomes 1, (e.g. $xx...xmm$ becomes 00...011) and convert that binary number into a decimal i (e.g. 00...011 becomes 3). L_i gives us the label of the layer in $PH_n^k(m)$ that contains that vertex (e.g. $xx...xmm$ is in L_3). This gives us a label for each layer in $PH_n^k(m)$.

It is easy to see that $B_{0,1,3}$ is isomorphic to L_0 when peg 2 is fixed (See figures 3(a) and 3(b)).

There are k ways to partition H_n^k, and each way corresponds to fixing a different peg. Figure 3(b) shows only one of the four ways to partition H_2^4.

Theorem 3.7. *There are 2^n layers in $PH_n^4(m)$.*

Proof. Each unique configuration of disks on peg m corresponds to a layers of $PH_n^4(m)$. The total number of unique configurations of disks on peg m is equal to

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

□

Proof. Another proof is that each vertex in $B_{x,m}$ where $x \neq m$ corresponds to a unique configuration of disks on peg m. There are 2^n vertices in $B_{x,m}$, so the total number of unique configurations of disks on peg m is 2^n. □

Also, since each vertex in $B_{x,m}$ where $x \neq m$ is in a layer and there is a unique vertex in $B_{x,m}$ in each layer (proved in 3.5), there are an equal number of layers in H_n^4 and number of vertices in $B_{x,m}$. Thus, layers are labeled $L_0, L_1, ...L_{2^n-1}$ as we can see in figure 3(b).

Theorem 3.8. *Given an arbitrary vertex in $PH_n^k(m)$, we can always find in which layer this vertex lies. In the label of the vertex, each instance of a number that is not m becomes 0, and each instance of m becomes 1. We convert this binary number into a decimal i and the vertex is in L_i in $PH_n^k(m)$.*

For example, in $PH_5^4(2)$, 12302 is in the (01001 base 2)th layer, i.e. P_{17}.

Proof. Given two vertices in H_n^k labeled $a = a_1a_2...a_{n-1}$ and $b = b_1b_2...b_{n-1}$, if $i = j \;\; \forall\, i, j$ such that $a_i = b_j = m$, then vertices a and b are in the same layer in $PH_n^k(m)$. Thus, we may generalize 3.6 into 3.8. □

Theorem 3.9. *If we let decimal x be rewritten as $\sum_{i=0}^{n} a_i 2^i$ where $a_i \in \{0, 1\}$ (in other words, the binary expression of x) and let $y = \sum_{i=0}^{n} a_i$, then in $PH_n^4(m)$, layer L_x is isomorphic to H_{n-y}^3.*

Proof. y is the number of 1s in the binary representation of x, and that gives us the the number of m's in any vertex in layer L_x (cf.). Thus, y gives us the number of disks on peg m, and $n - y$ gives us the number of disks on the remaining three pegs. □

The Frame-Stewart algorithm tells us after moving the smallest $n - l_n$ disks from peg 0 to peg 2, to travel along the layer $L_{2^{n-l_n}}$, which is a $H^3_{l_n}$ graph. Thus, the three steps of the Frame-Stewart algorithm can be thought of as three graphs, $H^4_{n-l_n}$, $H^3_{l_n}$ and $H^4_{n-l_n}$, respectively, and the Frame-Stewart solution splices these graphs together.

Proposition 3.10. *If the Frame-Stewart algorithm gives the optimal solution from one perfect state to another on 4 pegs, then the vertex on $B_{x,y}$ that is the shortest distance away from a perfect state, say* $\mathbf{z}$*, in H^4_n lies on layer $L_{2^{n-l_{n+1}}}$ in $PH^k_n(m)$ where $x \neq y \neq z$ and either $x = m$ or $y = m$.*

Proof. We assume that the shortest path between two perfect states, say $\mathbf{0}$ and $\mathbf{1}$, only travels along one bridge. Since the only bridges between [0] and [1] lie between $0B_{2,3}$ and $1B_{2,3}$ and since the number of moves made in [0] are equal to the number of moves made in [1] (say x moves), the shortest path takes $2x + 1$ moves. Then, the shortest path between perfect states $\mathbf{0}$ and $\mathbf{1}$ must minimize x. In order to minimize x, the shortest path between perfect states $\mathbf{0}$ and $\mathbf{1}$ must visit the vertex on $0B_{2,3}$ which is the shortest distance away from $\mathbf{0}$. The Frame-Stewart algorithm tells us to travel along the layer $L_{2^{n-l_n}}$. The bridge we take between [0] and [1] must in the layer $L_{2^{n-l_n}}$ and thus the bridge must be between the vertices in $0B_{2,3}$ and $1B_{2,3}$ which are in the layer $L_{2^{n-l_n}}$. Thus the vertex in $0B_{2,3}$ which is the shortest distance away from $\mathbf{0}$ lies in the layer $L_{2^{n-l_n}}$. Similarly, the vertex in $B_{2,3}$ which is the shortest distance away from $\mathbf{0}$ lies in the layer $L_{2^{n-l_{n+1}}}$. □

References

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