MT 430 Intro to Number Theory
PROBLEM SET 7

Due Thursday 4/18

Problem 1. Given that $\sqrt{18} = \langle 4, 4, 8 \rangle$, find the least positive solution of $x^2 - 18y^2 = -1$ (if any) and of $x^2 - 18y^2 = 1$.

Problem 2. Given $\sqrt{29} = \langle 5, 2, 1, 1, 2, 10 \rangle$, find the least positive solution of $x^2 - 29y^2 = -1$ (if any) and of $x^2 - 29y^2 = 1$.

Problem 3. Prove that there are infinitely many positive integers $n$ such that $\frac{n(n + 1)}{2}$ is a perfect square.

Problem 4. Prove that there are infinitely many positive integers $n$ such that $n^2 + (n + 1)^2$ is a perfect square.

Problem 5. Prove that $x^2 - dy^2 = -1$ has no integer solutions if $d \equiv 3 \pmod{4}$.

Problem 6 (Harder). Let $d$ be a positive integer, which is not a perfect square. Suppose $N$ is such that $x^2 - dy^2 = N$ has one integer solution. Prove that $x^2 - dy^2 = N$ has in fact infinitely many solutions.

Definition. Integer solutions of the Diophantine equation

$$x^2 + y^2 = z^2$$

are called Pythagorean triples. Note that if $(x, y, z)$ is a Pythagorean triple and an integer $d$ divides any two of $(x, y, z)$ then it divides all three of them. Consequently, one is most interested in Pythagorean triples $(x, y, z)$ such that $\gcd(x, y) = \gcd(x, z) = \gcd(y, z) = 1$. Such triples are called primitive.

In the next two exercises, you will find all primitive Pythagorean triples.

Problem 7. Suppose $(x, y, z)$ is a primitive Pythagorean triple. Show that exactly one of $x$ and $y$ is odd and the other is even.

Problem 8. Suppose $(x, y, z)$ is a primitive Pythagorean triple with $x$ even. Prove that $z - y = 2s^2$ and $z + y = 2t^2$ for some coprime integers $s$ and $t$.

Conclude that all primitive Pythagorean triples $(x, y, z)$ are of the form $(x, y, z) = (2st, t^2 - s^2, t^2 + s^2)$ for some coprime integers $s$ and $t$. 

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