MT 806 Algebra I
PROBLEM SET 1

Due Friday 9/14

Problem 1. Let $G$ be a group. Suppose that $G/Z(G)$ is cyclic. Prove that $G$ is abelian.

Problem 2. Let $H ≅ \langle (1234) \rangle ≤ S_n$, where $n ≥ 4$. Compute the normalizer of $H$ in $S_n$.

Problem 3. Given an action of a group $G$ on a set $A$, we define $σ_g: A → A$ by $σ_g(a) = g · a$. Verify that the map $φ: G → S_A$ given by $g ↦ σ_g$ is a group homomorphism.

Problem 4. Suppose a group $G$ acts on a set $X$. Show that $G · x = gG · x g^{-1}$ for every $x ∈ X$ and $g ∈ G$. In other words, the stabilizers of elements in the same orbit are all conjugates.

Problem 5. Compute the number of ways to color the vertices of a cube into 3 colors. Two colorings are equivalent if obtained from one another by a rotation (but not a flip!) in $\mathbb{R}^3$.

Problem 6. Suppose a group $G$ acts on a set $X$ and $H ≤ G$ is a subgroup. Let

$$X^H := \{ x ∈ X \mid g · x = x, \forall g ∈ H \}$$

be the fixed-point set of $H$.

(a) Prove that $N_G(H)$ preserves $X^H$.

(b) Prove that the center of $S_n$ is trivial for $n ≥ 3$.

Problem 7. Suppose $H$ is a subgroup of $S_n$ of index $n$. Prove that $H ≅ S_{n-1}$.

Problem 8. Suppose $n$ is such that $S_n$ has a transitive subgroup\(^1\) of index $n$. Prove that $S_n$ has an outer automorphism\(^2\). Prove that $S_6$ has an outer automorphism.

---

\(^1\)A subgroup of $S_X$ is called transitive if it acts transitively on $X$.

\(^2\)A group automorphism is called outer if it is not inner.