Problem 1. Let $I$ and $J$ be ideals of $R$.

1. Prove that $I + J$ is the smallest ideal containing $I$ and $J$.
2. Prove that $IJ$ is an ideal contained in $I \cap J$.
3. Give an example when $IJ \neq I \cap J$.
4. Prove that if $R$ is commutative and $I + J = R$, then $IJ = I \cap J$.

Problem 2. Let $d$ be a square-free integer and $\mathcal{O} = \mathbb{Z}[\omega]$ the ring of integers in the quadratic field $\mathbb{Q}(\sqrt{d})$. For any positive integer $f$, let $\mathcal{O}_f = \{a + bf\omega \mid a, b \in \mathbb{Z}\}$. Then $\mathcal{O}_f$ is a unitary subring of $\mathcal{O}$. Prove that $[\mathcal{O} : \mathcal{O}_f] = f$ (index of an abelian additive subgroup). Conversely, prove that if $R \subset \mathcal{O}$ is a unitary subring of $\mathcal{O}$ such that $[\mathcal{O} : R] = f$, then $R = \mathcal{O}_f$.

Problem 3. A commutative ring with 1 is called a local ring if it has a unique maximal ideal. Prove that if $R$ is a local ring with maximal ideal $M$ then every element of $R - M$ is a unit. Conversely, prove that if $R$ is a commutative ring with 1 in which the set of nonunits forms an ideal $M$, then $R$ is a local ring with unique maximal ideal $M$.

Problem 4. Let $K$ be a field. A discrete valuation of $K$ is a function $\nu : K^\times \to \mathbb{Z}$ satisfying

1. $\nu(ab) = \nu(a) + \nu(b)$.
2. $\nu$ is surjective.
3. $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$ for all $x, y \in K^\times$ with $x + y \neq 0$.

N.B. We can also set $\nu(0) = \infty$ to extend $\nu : K \to \mathbb{Z} \cup \infty$.

The set $R = \{x \in K^\times \mid \nu(x) \geq 0\} \cup \{0\}$ is called the valuation ring of $\nu$.

1. Prove that $R$ is a subring of $R$ which contains 1.
2. Prove for $x \in K^\times$ that either $x$ or $x^{-1}$ is in $R$.
3. Prove that an element $x$ of $R$ is a unit if and only if $\nu(x) = 0$.
4. Prove that $R$ is a PID.
5. Prove that $R$ is a local ring (in particular, it has unique maximal ideal).
6. Prove that $R$ has a unique non-zero prime ideal.
7. Prove that $R$ has a unique irreducible element (up to associates).

N.B. Valuation rings of fields with discrete valuation are called discrete valuation rings.

Problem 5. Consider the quadratic field $\mathbb{Q}(\sqrt{d})$ with an associated quadratic integer ring $\mathcal{O}$ and field norm $N$, $N(z) = z\bar{z}$.

1. For $d = -2, -3, -7, -11$ prove that $\mathcal{O}$ is a Euclidean domain with respect to $N$.
2. For $d = 2$, prove that $\mathcal{O}$ is a Euclidean domain with respect to the absolute value of $N$.

Problem 6. Prove that $\mathbb{Z}[\sqrt{-n}]$ is not a UFD for $n > 3$. Conclude that the ring of integers in $\mathbb{Q}(\sqrt{d})$ is not a UFD for $d < -3, d \equiv 2, 3 \pmod{4}$.