**Problem 1.** Prove that a ring homomorphism \( \phi : R \to S \) is finite if and only if \( \phi \) is finite type and integral.

**Remark.** Note that \( \phi \) is finite (resp., finite type, resp., integral) if and only if the ring extension \( \phi(R) \subset S \) is. Hence in this problem we can pass to the case of a ring extension \( R \subset S \), as is customary.

**Proof.** Suppose \( \phi \) is finite. Let \( x_1, \ldots, x_n \) be the generators of \( S \) as an \( R \)-module. Clearly, \( x_1, \ldots, x_n \) generate \( S \) as an \( R \)-algebra. Hence \( \phi \) is finite type. By Proposition 1.1.4 (3), every element of \( S \) is integral over \( R \). Hence \( \phi \) is integral.

Suppose \( \phi \) is finite type and integral. Let \( x_1, \ldots, x_n \) generate \( S \) as an \( R \)-algebra. Since every element of \( S \) is integral over \( R \), we have that \( x_1 \) is integral over \( R \) and, for each \( i = 2, \ldots, n \), the element \( x_i \) is integral over \( R[x_1, \ldots, x_{i-1}] \). It follows by Proposition 1.1.4 (2) that

\[
R[x_1, \ldots, x_{i-1}] \subset R[x_1, \ldots, x_i]
\]

is a finite ring extension for each \( i = 1, \ldots, n \). That \( \phi \) is finite now follows from the following easy lemma:

**Lemma.** A composition of finite ring extensions is finite.

**Proof of lemma.** Suppose \( R \subset S \subset T \) are ring extensions such that \( S \) is generated by \( s_1, \ldots, s_n \) as \( R \)-module and \( T \) is generated by \( t_1, \ldots, t_m \) as \( S \)-module. Then it is easy to see that \( \{s_it_j\}_{i,j=1}^{n,m} \) generate \( T \) as \( R \)-module.

**Problem 2.** Let \( R \) be a ring. Prove that every ideal \( I \) of \( R \) has at least one minimal prime. Moreover, given a prime \( p \) containing \( I \), prove that there is a minimal prime of \( I \) that lies in \( p \).

**Proof.** Every ideal is contained in a maximal ideal, so it suffices to prove the second statement. Suppose \( I \) is contained in a prime \( p \). Consider the set \( \Sigma \) of all prime ideals \( q \) such that

\[
I \subset q \subset p.
\]

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We endow \(\Sigma\) with a partial order given by the relation \(\supset\). Then the existence of a minimal prime of \(I\) lying inside \(p\) follows from the Zorn’s Lemma once the following useful statement is established:

**Lemma.** Suppose \(\{q_\lambda\}_{\lambda \in S}\) is a chain of prime ideals (i.e., any two ideals in \(S\) are comparable). Then the intersection

\[
J := \bigcap_{\lambda \in S} q_\lambda
\]

is a prime ideal.

*Proof of lemma.* Suppose \(a, b \notin J\). Then there exists \(k \in S\) such that \(a \notin q_k\) and there exists \(\ell \in S\) such that \(b \notin q_\ell\). Suppose \(q_k \supset q_\ell\). Then \(a, b \notin q_\ell\) and so \(ab \notin q_\ell\) by the primeness of \(q_\ell\). We conclude that \(ab \notin J\). This shows that \(J\) is a prime ideal. \(\square\)

**Problem 3.** Prove that an \(R\)-module \(M\) is Noetherian if and only if every submodule of \(M\) is finitely generated.

*Proof.* By definition, \(M\) is Noetherian if and only if the submodules of \(M\) satisfy the ascending chain condition (the a.c.c.) if and only if every non-empty set of submodules of \(M\) has a maximal element. We thus need to prove the following lemma:

**Lemma.**

1. If every non-empty set of submodules of \(M\) has a maximal element, then every submodule of \(M\) is finitely generated.
2. If every submodule of \(M\) is finitely generated, then the submodules of \(M\) satisfy the a.c.c.

*Proof of lemma.*

1) Suppose every non-empty set of submodules has a maximal element. Let \(N\) be a submodule of \(M\). Suppose \(N\) is not finitely generated. Let \(N'\) be the maximal among finitely generated submodules of \(N\). Then \(N' \neq N\). Take \(x \in N \setminus N'\). Then \(N' + Rx\) is a finitely generated submodule of \(N\) that properly contains \(N'\). A contradiction!

2) Suppose that all submodules of \(M\) are finitely generated. Let

\[
M_1 \subset M_2 \subset \cdots
\]

be an ascending chain of submodules of \(M\). By assumption,

\[
\bigcup_{i \geq 1} M_i
\]
is a finitely generated module. Let $x_1, \ldots, x_d$ be its generators. For each $j = 1, \ldots, d$, there exists $i_j$ such that $x_j \in M_{i_j}$. It is then clear that for $n = \max\{i_j \mid j = 1, \ldots, d\}$, we have $M_n = M_{n+1} = \cdots$, i.e., the chain is stationary.

□

Problem 4. For a fixed prime $p$, let $\mathbb{Z}_p = \left\{ \frac{m}{p^n} \mid m, n \in \mathbb{Z} \right\} \subset \mathbb{Q}$ and $W = \mathbb{Z}_p/\mathbb{Z}$. Prove that $W$ is an Artinian, but not Noetherian, $\mathbb{Z}$-module.

Remark. Recall that a $\mathbb{Z}$-module is the same thing as an abelian group.

Remark. Note that $\mathbb{Z}_p = \mathbb{Z} \left[ \frac{1}{p} \right]$ is the ring of fractions of $\mathbb{Z}$ with respect to $\{p^n\}_{n \geq 0}$.

Proof. The existence of a non-stationary ascending chain of subgroups in $W$ proves that $W$ is not a Noetherian $\mathbb{Z}$-module:

$$\mathbb{Z} \frac{1}{p} \subset \mathbb{Z} \frac{1}{p^2} \subset \cdots .$$

Before we prove that $W$ is Artinian, we establish a structure result about subgroups of $W$:

Lemma. Every proper subgroup of $W = \mathbb{Z}_p/\mathbb{Z}$ is cyclic, generated by an element of the form $\frac{1}{p^n}$.

Proof of lemma. Let $H$ be a subgroup of $W$. Let $\Gamma$ be the set of positive integers $m$ such that $H$ contains an element of the form $\frac{a}{p^m}$ with $\gcd(a, p) = 1$. Clearly, if $m \in \Gamma$, then $m - 1 \in \Gamma$. Hence either $\Gamma = \mathbb{Z}_{>0}$, or $\Gamma = \{1, \ldots, n\}$ for some $n$.

Note that if $\frac{a}{p^m} \in H$ for some $a$ with $\gcd(a, p) = 1$, then $ai + j p^n = 1$ for some integers $i$ and $j$. We then have that

$$\frac{1}{p^m} = i \frac{a}{p^m} + j = i \frac{a}{p^m} \in H .$$

In particular, if $\Gamma = \mathbb{Z}$, then $H = W$, and if $\Gamma = \{1, \ldots, n\}$, then $H = \left\langle \frac{1}{p^n} \right\rangle$. □

By the above lemma, every proper $\mathbb{Z}$-submodule of $W$ is of the form $\mathbb{Z} \frac{1}{p^n}$, where $n$ is a positive integer. It is now easy to see that every descending chain
of submodules in $W$ is stationary. Indeed, this follows from

$$\mathbb{Z} \frac{1}{p^n} \supset \mathbb{Z} \frac{1}{p^m} \iff n \geq m$$

and the fact that every non-increasing sequence of positive integers is stationary. □

**Problem 5.** Suppose $R$ is a Noetherian ring. Then an $R$-module $M$ is Noetherian if and only if $M$ is finitely generated.

**Proof.** Recall that an $R$-module is finitely generated if it is a quotient of a finite free module $R^n$, for some positive integer $n$.

Suppose $R$ is a Noetherian ring. Then $R$ is a Noetherian $R$-module and so $R^n = R \oplus \cdots \oplus R$ is a Noetherian $R$-module by Lemma 1.2.8. Quotients of Noetherian modules are Noetherian by the same lemma. It follows that any finitely generated $R$-module is Noetherian.

The converse holds for any ring: A Noetherian module is finitely generated by Problem 3 above. □

**Problem 6.** Find an example of the following:

1. A sub-algebra of a finitely generated algebra that is not finitely generated.

2. A subring of a Noetherian ring that is not Noetherian.

**Proof.** (1) Analogously to the case of $R$-modules (cf. Problem 3), we have the following result:

**Lemma.** For an $R$-algebra $S$, the following are equivalent:

1. Every $R$-subalgebra of $S$ is a finitely generated $R$-algebra.
2. Every non-empty set of $R$-subalgebras of $S$ has a maximal element.
3. Every ascending chain of $R$-subalgebras of $S$ is stationary.

We will leave the proof of the above lemma as an exercise and only note that a non-stationary chain of $R$-subalgebras

$$S_1 \subseteq S_2 \subseteq \cdots$$

of $S$ defines a non-finitely generated $R$-algebra $\bigcup_{i \geq 1} S_i$.

With the above as an inspiration, we take $R$ to be any non-zero ring and

$$S = R[x, xy, xy^2, \ldots]$$

to be the subalgebra of $R[x, y]$ generated by the elements $\{xy^n\}_{n \geq 0}$. Clearly, we have an ascending chain of $R$-subalgebras

$$R \subset R[x] \subset R[x, xy] \subset R[x, xy, xy^2] \subset \cdots.$$
This chain is non-stationary because \( xy^{n+1} \notin R[x, xy, \ldots, xy^n] \). (Any degree 2 or more polynomial in \( x, xy, \ldots, xy^n \) is divisible by \( x^2 \), and no linear polynomial in \( x, xy, \ldots, xy^n \) can be equal to \( xy^{n+1} \).)

It follows that \( S \) is a non-finitely generated \( R \)-subalgebra of a finitely generated algebra \( R[x, y] \).

(2) Let \( R \) be some non-Noetherian domain, e.g., \( R = k[x_1, x_2, \ldots] \) (infinitely many variables). We have that \( R \) is a subring of the field \( \text{Frac}(R) \), which is trivially a Noetherian ring. \( \square \)

**Problem 7.** Let \( A \) be a Noetherian ring and \( B \) be a finitely generated \( A \)-algebra. Let \( G \) be a finite group of \( A \)-automorphisms of \( B \). Let \( B^G \) be the ring of invariants:

\[
B^G = \{ b \in B \mid \phi(b) = b \text{ for every } \phi \in G \}.
\]

Prove that \( B^G \) is a finitely generated \( A \)-algebra.

**Proof.** As customary, we can assume that \( A \subset B \) is a ring extension.

In the chain of ring extensions \( A \subset B^G \subset B \), we have that \( B \) is a finitely generated \( A \)-algebra. The fact that \( B^G \) is a finitely generated \( A \)-algebra will follow from Proposition 1.3.13 (Artin-Tate) at once as soon as we prove the following result:

**Lemma.** The ring extension \( B^G \subset B \) is integral.

**Proof of lemma.** Take \( b \in B \) and consider the following polynomial in \( t \):

\[
p(t) = \prod_{h \in G} (t - h \cdot b).
\]

Clearly, \( p(b) = 0 \). Also, \( p(t) \) is clearly a \( G \)-invariant polynomial. Indeed, the action of \( G \) on \( B \) extends in a natural way to an action of \( G \) on \( B[t] \), where \( G \) acts trivially on \( t \). The action of \( G \) on \( B[t] \) respects the ring structure, and hence

\[
g \cdot p(t) = \prod_{h \in G} (t - g \cdot (h \cdot b)) = \prod_{h \in G} (t - h \cdot b) = p(t).
\]

It follows that all the coefficients of \( p(t) \) are \( G \)-invariant, and so lie in \( B^G \). Since \( p(t) \) is monic, we conclude that \( b \) is integral over \( B^G \). This finishes the proof. \( \square \)

**Problem 8.** Let \( k \) be a field of characteristic \( \neq 2 \). Let \( R = k[x, y] \). Let \( G \cong \mathbb{Z}_2 \) be a group of \( k \)-automorphisms of \( R \), where the generator of \( G \) acts by \( x \mapsto -x \) and \( y \mapsto -y \). Compute \( R^G \) and find a set of generators of \( R^G \) as a \( k \)-algebra.
Proof. We have that \( G = \langle \varepsilon \rangle \), where \( \varepsilon^2 = 1 \). It is clear that (all sums below are finite):

\[
\varepsilon \cdot \left( \sum a_{ij} x^i y^j \right) = \sum (-1)^{i+j} a_{ij} x^i y^j.
\]

It follows that

\[
R^G = \left\{ \sum a_{ij} x^i y^j \mid i + j \text{ is even whenever } a_{ij} \neq 0 \right\}.
\]

Obviously, \( x^2, y^2, xy \in R^G \); and every element of the form \( x^i y^j \), where \( i + j \) is even, can be written as a product of elements in \( \{ x^2, y^2, xy \} \). It follows that

\[
R^G = k[x^2, y^2, xy].
\]

\[\square\]

Problem 9. Let \( R \subset S \) be rings and assume that \( R \) is a summand of \( S \) as an \( R \)-module, that is, there exists an \( R \)-module homomorphism \( \pi : S \to R \) such that \( \pi(x) = x \) for every \( x \in R \). Prove that if \( S \) is a Noetherian ring, then \( R \) is also a Noetherian ring.

Proof. Suppose \( I \subset R \) is an ideal of \( R \). Since \( S \) is a Noetherian ring, the extended ideal \( IS \) has finitely many generators. So suppose \( IS = (x_1, \ldots, x_n) \), where \( x_i \in S \). Note that, by the definition of \( IS \), every \( x_i \) can be written as a finite linear combination of the elements in \( I \) with coefficients in \( S \). Let \( y_1, \ldots, y_m \) be all the elements of \( I \) that appear in such expansions. Then for every \( i = 1, \ldots, n \), we have

\[
x_i = \sum_{j=1}^m s_{ij} y_j, \quad \text{for some } s_{ij} \in S.
\]

Since \( x_1, \ldots, x_n \) generate \( IS \) as an \( S \)-module, it follows that \( y_1, \ldots, y_m \in I \) also generate \( IS \) as an \( S \)-module.

I claim now that \( y_1, \ldots, y_m \) generate \( I \) as an \( R \)-module, hence as an ideal in \( R \). Suppose \( a \in I \). Then \( a \in IS \) and so we can write

\[
a = \sum_{j=1}^m r_j y_j = \sum_{j=1}^m y_j r_j, \quad \text{for some } r_j \in S.
\]

Applying \( \pi \) and using the fact that \( \pi \) is an \( R \)-module homomorphism, we obtain

\[
a = \pi(a) = \sum_{j=1}^m y_j \pi(r_j) = \sum_{j=1}^m \pi(r_j) y_j \in (y_1, \ldots, y_m).
\]

This finishes the proof. \[\square\]
**Problem 10.** Let \( R = R_0 \oplus R_1 \oplus \cdots \) be a graded ring. Prove that the following are equivalent:

1. \( R \) is Noetherian.
2. \( R_0 \) is Noetherian and the irrelevant ideal \( I_+ := R_1 \oplus R_2 \oplus \cdots \) is finitely generated.
3. \( R_0 \) is Noetherian and \( R \) is a finitely generated \( R_0 \)-algebra.

**Proof.**

(1) \( \Rightarrow \) (2): Suppose \( R \) is Noetherian. Then every ideal of \( R \) is finitely generated. In particular, \( I_+ \) is finitely generated. Note that \( R_0 \cong R/I_+ \). Hence \( R_0 \) is Noetherian.

(3) \( \Rightarrow \) (1): This follows from the Hilbert Basis Theorem (Theorem 1.3.10).

(2) \( \Rightarrow \) (3): It suffices to establish the following lemma, where no assumption on \( R_0 \) is made.

**Lemma.** Suppose \( R = R_0 \oplus R_1 \oplus \cdots \) is a graded ring. If the irrelevant ideal \( I_+ = R_1 \oplus R_2 \oplus \cdots \) is finitely generated, then \( R \) is a finitely generated \( R_0 \)-algebra.

**Proof of lemma.** Suppose \( x_1, x_2, \ldots, x_n \) are homogeneous elements generating the ideal \( I_+ \). Set \( d_i = \deg(x_i) \) for \( i = 1, \ldots, n \).

We are going to prove that \( R = R_0[x_1, \ldots, x_n] \). We do so by proving by induction on \( d \) that \( R_d \subset R_0[x_1, \ldots, x_n] \). The case of \( d = 0 \) is the base case. Suppose we know that \( R_0 \oplus \cdots \oplus R_{d-1} \subset R_0[x_1, \ldots, x_n] \) for some \( d \geq 1 \). Take \( f \in R_d \). Then \( f \in I_+ \) and so can be written as

\[
f = \sum_{i=1}^{n} a_i x_i,
\]

where \( a_i \in R \). Since \( f \) and \( x_i \)'s are homogeneous, we must have that \( a_i \)'s are homogeneous; in fact, \( a_i \in R_{d-d_i} \). Since \( d_i \geq 1 \), we have that \( a_i \in R_0[x_1, \ldots, x_n] \) by the inductive assumption. We conclude that \( f \in R_0[x_1, \ldots, x_n] \).

### Footnote

[1] Starting from a finite set of not necessarily homogeneous generators of \( I_+ \), we obtain a finite set of homogeneous generators by taking the union of the homogeneous parts of all generators.