10.2. Problem Set 2 Solution

**Problem.** Prove the following statements.

1. The nilradical of a ring $R$ is the intersection of all prime ideals of $R$.
2. The radical of an ideal $I$ is the intersection of all prime ideals of $R$ containing $I$.

**Proof.** For the quotient homomorphism $\phi: R \rightarrow R/I$, we have

$$\phi^{-1}(\text{nilrad}(R/I)) = \{ f \in R \mid \phi(f)^n = 0 \text{ for some } n \in \mathbb{N} \}$$

$$= \{ f \in R \mid f^n \in I \text{ for some } n \in \mathbb{N} \} = \text{rad}(I).$$

Therefore, the second claim follows from the first one, which we proceed to prove.

The inclusion

$$\text{nilrad}(R) \subset \bigcap_{p \in \text{Spec } R} p$$

is obvious, for $f^n = 0$ implies that $f \in p$ for every prime $p$.

Suppose $f$ is not nilpotent. Then $S = \{ f^n \}_{n=0}^{\infty}$ is a multiplicative set not containing 0. Hence by Problem 2.6, there exists a prime ideal $p$ disjoint from $S$. (See below for details.) In particular, $f \notin p$. This finishes the proof.

**Problem.** Let $k$ be a field. Describe the contraction of the maximal ideal $(x - 1, y - 1)$ under the ring homomorphism $f: k[z_1, z_2, z_3] \rightarrow k[x, y]$ given by

$$z_1 \mapsto x^2, \quad z_2 \mapsto xy, \quad z_3 \mapsto y^2.$$

We will do this problem by proving a much more general statement:

**Proposition 10.2.1.** Let $R = k[z_1, \ldots, z_m]/I$ and $S = k[x_1, \ldots, x_n]/J$ be finitely generated algebras over a field $k$. Suppose $f: R \rightarrow S$ is a $k$-algebra homomorphism defined by $f(z_i) = \phi_i(x_1, \ldots, x_n)$ for some polynomials $\phi_i \in k[x_1, \ldots, x_n]$. (Note that $\phi_1, \ldots, \phi_m$ have to satisfy $F(\phi_1, \ldots, \phi_m) \in J$ for every $F \in I$.) Then for every $(a_1, \ldots, a_n) \in k^n$ such that $J \subset (x_1 - a_1, \ldots, x_n - a_n)$, the contraction of the induced maximal ideal $(x_1 - a_1, \ldots, x_n - a_n) \subset S$ is a maximal ideal

$$(z_1 - \phi_1(a_1, \ldots, a_n), \ldots, z_n - \phi_n(a_1, \ldots, a_n)) \subset R.$$

**Lemma 10.2.2.** Every $k$-algebra homomorphism

$$\text{ev}_a: k[x_1, \ldots, x_n] \rightarrow k,$$

is defined by $\text{ev}_a(x_i) = a_i$ for some $a = (a_1, \ldots, a_n) \in k^n$ and we have $\ker(\text{ev}_a) = (x_1 - a_1, \ldots, x_n - a_n)$. In particular, $(x_1 - a_1, \ldots, x_n - a_n)$ is a maximal ideal for every $(a_1, \ldots, a_n) \in k^n$. 

\[\square\]
Proof. This follows from two observations. The first is that $ev_a(F(x_1, \ldots, x_n)) = F(a_1, \ldots, a_n)$ for every $F \in k[x_1, \ldots, x_n]$. The second is that

\[ F(x_1, \ldots, x_n) = F(a_1, \ldots, a_n) \mod (x_1 - a_1, \ldots, x_n - a_n). \]

N.B. The above follows from the existence of the Taylor expansion of $F(x_1, \ldots, x_n)$ around $(a_1, \ldots, a_n)$. \qed

N.B. A $k$-algebra homomorphism from a finitely generated $k$-algebras to $k$ is called an **evaluation homomorphism**. Lemma 10.2.2 completely describes such homomorphisms.

**Proof.** By Lemma 10.2.2, we have that $(x_1 - a_1, \ldots, x_n - a_n) \subset S$ is the kernel of some evaluation homomorphism $ev_a : S \to k$ defined by $ev_a(x_i) = a_i$. Consider now the composition homomorphism

\[ R \xrightarrow{f} S \xrightarrow{ev} k. \]

Then $ev \circ f$ is the evaluation homomorphism from $R$ to $k$ defined by $z_i \mapsto \phi_i(a_1, \ldots, a_n)$. In particular, $(z_1 - \phi_1(a_1, \ldots, a_n), \ldots, z_n - \phi_n(a_1, \ldots, a_n)) = \ker(ev \circ f)$, again by Lemma 10.2.2. This finishes the proof. \qed

**Problem.** Suppose $k$ is a field (not necessarily algebraically closed; the case of algebraically closed field was done in class). For a homomorphism of finitely generated $k$-algebras $R \to S$, prove that the contraction of a maximal ideal in $S$ is a maximal ideal in $R$.

**Proof.** Let $n \subset S$ be a maximal ideal and $K = S/n$ be the corresponding quotient field. Then $k \subset K$ is a finitely generated field extension, hence $k \subset K$ is a finite field extension by Weak Nullstellensatz (Proposition 1.4.1). Let $m := n \cap R$ be the contraction of $n$ in $R$. Then we have a sequence of ring extensions:

\[ k \subset R/m \subset K. \]

Since $k \subset K$ is a finite ring extension, we must have that $R/m \subset K$ is a finite ring extension. But both of these rings are domains and $K$ is a field. It follows that $R/m$ is a field by Proposition 3.6.3. \qed

**Problem.** Finish the proof of Proposition 2.0.6 from the lecture notes by showing that for an irreducible affine variety $X \subset k^n$, the ideal of polynomials vanishing on $X$ is prime.

**Proof.** Suppose that $X$ is an irreducible affine variety but the ideal $I := I(X)$ is not prime. Then there exist $f, g \in k[x_1, \ldots, x_n]$ such that $f, g \notin I(X)$, but $fg \in I(X)$. Let $X_1 = Z(I, f) \subset Z(I(X)) = X$ and $X_2 = Z(I, g) \subset Z(I(X)) = X$ be algebraic sets. But $X_1 \cup X_2 = Z((I, f)(I, g)) \supset Z(I) = X$. It follows that $X_1 \cup X_2 = X$. But because $X$ is irreducible, we conclude that either $X_1 = X$ and $X_2 = X$. Suppose $X_1 = X$. Then $\text{rad}(I, f) = \text{rad}(I) = I$. (Note that $I$ is automatically radical.) But then $f \in I$. A contradiction! \qed
**Problem.** Let $k$ be a field. For integers $1 \leq m < n$, consider the matrix

\[
M_{m,n} := \begin{pmatrix}
1 & x_1 & x_2 & \cdots & x_{m-1} & x_m \\
x_1 & x_2 & x_3 & \cdots & x_m & x_{m+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
x_{n-m} & \cdots & \cdots & \cdots & x_{n-1} & x_n
\end{pmatrix}.
\]

Let $I_{n,m}$ be the ideal in $k[x_1, \ldots, x_n]$ generated by the $2 \times 2$ minors of $M_{m,n}$. Prove that

\[
Z(I_{n,m}) = \{(t, t^2, t^3, \ldots, t^n) \mid t \in k\} \subset k^n.
\]

**Proof.** Note that for $i \leq m$ and $j \leq n - m$, the polynomial $x_{i+j} - x_i x_j$ belong to $I_{n,m}$. Suppose $(a_1, \ldots, a_n) \in Z(I_{n,m})$. Then we must have $a_{i+j} = a_i a_j$ for all $i \leq m$ and $j \leq n - m$. It follows that $a_r = a_1^r$ for every $1 \leq r \leq n$. \(\square\)

**Problem.** For any ideal $I \subset R$ and a multiplicative set $S$ disjoint from $I$, prove that there exists a prime ideal $p \supset I$ disjoint from $S$.

**Proof.** Let $\overline{R} := R/I$ and let $\overline{S}$ be the image of $S$ in $\overline{R}$. Then $\overline{S}$ is a multiplicative set in $\overline{R}$ that does not contain zero. In particular, the ring $\overline{S}^{-1}\overline{R}$ is not a zero ring. Let $q$ be a prime ideal in $\overline{S}^{-1}\overline{R}$. Then the contraction of $q$ to $R$ is a prime ideal avoiding $S$ and containing $I$.

(Alternatively, one can apply Zorn’s lemma to the set of the ideals containing $I$ and disjoint from $S$.) \(\square\)

**Problem.** Let $p$ be a prime ideal of a ring $R$. Prove that the residue field of $R_p$ is the fraction field of $R/p$:

\[
R_p/pR_p = \text{Frac}(R/p).
\]

**Proof.** Let $\phi : R \to R/p$ be the quotient homomorphism. There is an obvious ring homomorphism $\Phi : R_p \to \text{Frac}(R/p)$ that sends $a/s \in R_p$ to $\phi(a)/\phi(s)$. It is easy to see that $\Phi$ is well-defined (do this explicitly if you are still shaky with localization.) Clearly, $\Phi$ is surjective: Every element of $R/p$ can be written as $\phi(a)$ for $a \in R$ and every non-zero element of $R/p$ can be written as $\phi(s)$ for $s \notin p$.

Finally, $\Phi(a/s) = 0$ if and only if $\phi(a) = 0$ if and only if $a \in p$ if and only if $a/s \in pR_p$. Hence $\ker \Phi = pR_p$. We are done. \(\square\)

**Problem.** For an $R$-module $M$ and a multiplicative set $S \subset R$, prove that

\[
S^{-1}R \otimes_R M \cong S^{-1}M.
\]

**Proof.** We have an obvious $R$-bilinear map $S^{-1}R \times M \to S^{-1}M$ given by

\[
\left(\frac{r}{s}, m\right) \mapsto \frac{rm}{s}.
\]

By the universal property of the tensor product, this gives rise to an $R$-linear homomorphism

\[
f : S^{-1}R \otimes_R M \to S^{-1}M,
\]
that satisfies \(f \left( \frac{r}{s} \otimes m \right) = \frac{rm}{s} \). In fact, it is easy to see that \(f\) is also \(S^{-1}R\)-linear.

It is a tedious but straightforward exercise to check that

\[ g: S^{-1} \to S^{-1}R \otimes_R M, \quad g \left( \frac{m}{s} \right) = \frac{1}{s} \otimes m \]

is a well-defined \(R\)-module homomorphism (which is also an \(S^{-1}R\)-module homomorphism).

Clearly, \(f\) and \(g\) are inverses of each other and so define an \(S^{-1}R\)-module isomorphism

\[ S^{-1}R \otimes_R M \simeq S^{-1}M. \]

\[ \square \]

**Problem.** For a finitely generated \(R\)-module \(M\) and a multiplicative set \(S \subset R\), prove that \(S^{-1}M = 0\) if and only if there exists \(s \in S\) such that \(sM = 0\).

**Proof.** If \(sM = 0\) for some \(s \in S\), then for every element \(x = m/t \in S^{-1}M\), we have

\[ x = \frac{m}{t} = \frac{sm}{st} = 0 \in S^{-1}M. \]

Conversely, suppose \(S^{-1}M = 0\). Then for every \(m \in M\), there exists an element \(s_m \in S\) such that \(s_mm = 0\). (This follows by unwinding what it means for \(m = m/1\) to be zero in \(S^{-1}M\).)

By the assumption, we can choose finitely many generators \(m_1, \ldots, m_n\) of \(M\). Then it is easy to see that

\[ s := s_{m_1} \cdots s_{m_n} \]

satisfies \(sM = 0\). \(\square\)

**Problem.** Let \(R\) be a ring. Suppose that for each prime \(p\), the local ring \(R_p\) has no non-zero nilpotent elements. Prove that \(R\) has no non-zero nilpotent elements.

If each \(R_p\) is an integral domain, is \(R\) necessarily an integral domain?

**Proof.** Suppose \(r \in R\) is a nilpotent element. Then \(r = \frac{r}{1}\) remains nilpotent in \(R_p\). Thus we must have

\[ \frac{r}{1} = 0 \in R_p, \quad \text{for every prime } p. \]

Thus for each primes \(p\), there exists an element in \(R \setminus p\) that kills \(r\). In other words, the annihilator of \(r\)

\[ \text{Ann}(r) := \{ a \in R \mid ar = 0 \} \]

is not entirely contained in any of the prime ideals. Since \(\text{Ann}(r)\) is an ideal itself, we conclude that \(\text{Ann}(r) = R = (1)\) and so \(r = 0\).
Example: Consider a product of two domains, say \( R = S \times T \). Then a prime ideal in \( S \times T \) is of the form either \( S \times q \), where \( q \subset T \) is prime, or \( p \times T \), where \( p \subset S \) is prime. It is easy to see that
\[
(S \times T)_{S \times q} \simeq T_q.
\]
(Indeed, the complement of \( S \times q \) contains \( 0 \times 1 \), which allows to kill all non-zero elements of \( S \).) It follows that the localization of \( R \) is a domain for all primes of \( R \), but \( R \) is clearly not a domain.

\[\def\spec{\text{Spec}}\]

**Definition 10.2.4.** A ring \( R \) with no non-zero nilpotent elements (that is, satisfying \( \text{nilrad}(R) = (0) \)) is called **reduced**.

**Problem.** Let \( R \) be a ring and \( S \subset R \) a multiplicative set. For a ring homomorphism \( \phi: R \to S^{-1}R \), prove that
\[
\phi^\#(\spec S^{-1}R) = \bigcap_{f \in S} \spec R_f.
\]
Is the above set necessarily open?

**Proof.** By definition, \( \phi^\#(\spec S^{-1}R) \) is the set of prime ideals in \( R \) that are contractions of prime ideals in \( S^{-1}R \). The distinguished open \( \spec R_f \subset \spec R \) is defined to be the complement of \( V(f) \), and so consists of those prime ideals in \( R \) that do not contain \( f \).

A prime ideal \( p \subset R \) is a contraction of a prime ideal in \( S^{-1}R \) if and only if \( S \cap p = \emptyset \) if and only if \( f \notin p \) for every \( f \in S \) if and only if \( p \in \bigcap_{f \in S} \spec R_f \).

This finishes the proof.

If we take \( p \) to be a prime ideal and \( S = R \setminus p \), then \( \bigcap_{f \in S} \spec R_f \) is the set of all prime ideals contained in \( p \). Such a set is generally not open in Zariski topology. For example, if \( R = \mathbb{Z} \) and \( p = (p) \) for a prime \( p \), then \( \spec R_p = \{(0),(p)\} \), which is not open in \( \spec \mathbb{Z} \) because its complement is not closed. Another example would be \( R = k[x] \) and \( p = (x) \), etc. \(\square\)