1. (10 points) Let \( f(x, y, z) = xz + e^{y-x^2} \).
   a. (4 pts) Compute the gradient \( \nabla f \).
   b. (3 pts) Find the directional derivative \( D_{\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle} f(0, 0, 1) \).
   c. (3 pts) Find the unit vector pointing in the direction along which \( f(x, y, z) \) increases most rapidly at the point \((0, 0, 1)\).

Solution:
(a) We have \( \nabla f = \langle z - 2xe^{y-x^2}, e^{y-x^2}, x \rangle \) and so \( \nabla f(0, 0, 1) = \langle 1, 1, 0 \rangle \).
(b) Therefore
\[
D_{\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle} f(0, 0, 1) = \nabla f(0, 0, 1) \cdot \langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = \langle 1, 1, 0 \rangle \cdot \langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = 1/2.
\]
(c) The direction of the largest increase of \( f(x, y, z) \) at \((0, 0, 1)\) is \( \nabla f(0, 0, 1) = \langle 1, 1, 0 \rangle \) and the unit vector in this direction is \( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \).

2. (10 points)
   a. (5 pts) Find an equation of the tangent plane at \((1, 3)\) to the graph of \( f(x, y) = xy^2 - xy + 3x^3y \).

Solution: The equation of the tangent plane at \((1, 3)\) is:
\[
z - f(1, 3) = f_x(1, 3)(x - 1) + f_y(1, 3)(y - 3).
\]
We now compute
\[
f(1, 3) = 15 \quad f_x(1, 3) = 33 \quad f_y(1, 3) = 8.
\]
Therefore, the equation of the tangent plane at \((1, 3)\) is:
\[
z - 15 = 33(x - 1) + 9(y - 3).
\]
b. (5 pts) Find an equation of the tangent plane at the point (0, 3, −1) to the surface
\[ g(x, y, z) = ze^x + e^{z+1} + xy + y = 3. \]

Solution: We compute
\[ \nabla g(x, y, z) = \langle ze^x + y, x + 1, e^x + e^{z+1} \rangle. \]
The normal vector of the tangent plane at the point (0, 3, −1) is
\[ \nabla g(0, 3, −1) = \langle 2, 1, 2 \rangle, \]
so the tangent plane is
\[ 2x + (y - 3) + 2(z + 1) = 0. \]

3. (10 points) The surface contains curves \( r_1(t) = (t, t^2, t^3) \) and \( r_2(u) = (1, \sqrt{2}\sin(u), \sqrt{2}\cos(u)) \). Find the tangent plane to the surface at the point (1, 1, 1) and the line perpendicular to the tangent plane and passing through the origin (0, 0, 0).

Solution: We are going to use the fact that the tangent plane to a surface at a point \( P \) contains the tangent lines at \( P \) of all curves contained in the surface and passing through \( P \).

Here we have two curves contained in the surface and passing through \( P = (1,1,1) \). The tangent line of the first curve at \( P \) (corresponding to \( t = 1 \)) has direction vector \( \vec{v} = r_1'(1) = \langle 1, 2, 3 \rangle \) and the tangent line of the second curve at \( P \) (corresponding to \( u = \pi/4 \)) has direction vector \( \vec{w} = r_2'(\pi/4) = \langle 0, 1, -1 \rangle \). Since the normal vector of the tangent plane to the surface at \( P \) is perpendicular to these two vectors, we can take the normal vector of the tangent plane to be \( \vec{v} \times \vec{w} = \langle -5, 1, 1 \rangle \).

Finally, the equation of the tangent plane at \( P = (1,1,1) \) is
\[ -5(x - 1) + (y - 1) + (z - 1) = 0. \]
The line perpendicular to the tangent plane and passing through the origin (0, 0, 0) is
\[
\begin{align*}
x &= -5t \\
y &= t \\
z &= t
\end{align*}
\]
\[ \square \]

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4. (10 points) The function \( f(x, y) = x^2 + y^2 + xy + 9x \) has 1 critical point. Find it, and identify it as a local minimum, local maximum, or a saddle point.

Solution: To find critical points, we have to solve the system

\[
\begin{align*}
   f_x(x, y) &= 2x + y + 9 = 0 \\
   f_y(x, y) &= 2y + x = 0
\end{align*}
\]

Solving, we obtain \( y = 3 \) and \( x = -6 \). Hence \((x, y) = (-6, 3)\) is the only critical point.

Next we use the 2nd derivative test:

\[
\begin{align*}
   f_{xx}(-6, 3) &= 2 \\
   f_{yy}(-6, 3) &= 2 \\
   f_{xy}(-6, 3) &= 1
\end{align*}
\]

The determinant at \((-6, 3)\) is

\[
D(-6, 3) = f_{xx}(-6, 3)f_{yy}(-6, 3) - (f_{xy}(-6, 3))^2 = 4 - 1 = 3 > 0.
\]

Since \( f_{xx}(-6, 3) > 0 \), the 2nd derivative test says that \((-6, 3)\) is a local minimum. \(\square\)

5. (10 points) Consider the function \( f(x, y) = x^3y - 3xy^2 \). Find and classify all critical points of \( f \).

Solution: To find critical points, we have to solve the system

\[
\begin{align*}
   f_x(x, y) &= 3x^2y - 3y^2 = 3y(x^2 - y) = 0 \\
   f_y(x, y) &= x^3 - 6xy = x(x^2 - 6y) = 0
\end{align*}
\]

The only solution of this system is \((x, y) = (0, 0)\), hence \((0, 0)\) is the only critical point. Since \( f_{xx}(0, 0) = f_{yy}(0, 0) = f_{xy}(0, 0) \), the 2nd derivative test is inconclusive. However note that along \( y = x^2 \), the function is \( f(x, x^2) = x^5 - 3x^5 = -2x^5 \). Since \(-2x^5\) has neither maximum nor minimum at \( x = 0 \) \((-2x^5\) is strictly decreasing because \((-2x^5)' = -10x^4 \leq 0\). We conclude that \((0, 0)\) is a saddle point of \( f(x, y) \), i.e. \((0, 0)\) is neither local minimum nor local maximum. \(\square\)

6. (10 points) Find the maximum and minimum values of \( f(x, y) = x - 2y \) subject to the constraint \( \frac{x^2}{4} + y^2 = 2 \).

Solution: The method of Lagrange multipliers says that extrema of \( f(x, y) \) subject to the constraint are found among the solutions of \( \nabla f(x, y) = \lambda \nabla g(x, y) \), where \( g(x, y) = \frac{x^2}{4} + y^2 \). We rewrite this condition in the following form:

\[
\begin{align*}
   1 = f_x = \lambda g_x &= \lambda (x/2) \\
   -2 = f_y = \lambda g_y &= \lambda (2y) \\
   \frac{x^2}{4} + y^2 &= 2
\end{align*}
\]
or
\[
\begin{align*}
1 &= \lambda x/2 \\
-2 &= 2\lambda y \\
x^2/4 + y^2 &= 2
\end{align*}
\]
This gives \( x = 2/\lambda \) and \( y = -1/\lambda \). Plugging these into the constraint, we obtain:
\[
2 = \frac{(2/\lambda)^2}{4} + \frac{(-1/\lambda)^2}{2} = 2/\lambda^2,
\]
which gives \( \lambda = 1 \) or \( \lambda = -1 \). It follows that the two critical points of \( f(x, y) \) subject to the constraint are \((2, -1)\) and \((-2, 1)\).

We have \( f(2, -1) = 4 \) and \( f(-2, 1) = -4 \). Therefore, \( f(2, -1) = 4 \) is the maximum value of \( f(x, y) \) subject to the constraint \( x^2/4 + y^2 = 2 \); and \( f(-2, 1) = -4 \) is the minimum value of \( f(x, y) \) subject to the constraint \( x^2/4 + y^2 = 2 \).

\[\square\]

7. (10 points) Evaluate each of the following iterated integrals.

a. (5 pts)
\[
\int_0^1 \int_0^x e^{x^2+2y}dydx
\]
Solution:
\[
\int_0^1 \int_0^x e^{x^2+2y}dydx = \int_0^1 \left( \frac{1}{2} e^{x^2+2y} \right) \bigg|_0^x dx
\]
\[
= \int_0^1 \frac{1}{2}(e^{3x} - e^x)dx
\]
\[
= \left( \frac{e^{3x}}{6} - \frac{e^x}{2} \right) \bigg|_0^1 = \left( \frac{e^3}{6} - \frac{e}{2} \right) - \left( \frac{1}{6} - \frac{1}{2} \right) = \frac{e^3}{6} - \frac{e}{2} + 1/3.
\]

\[\square\]

b. (5 pts)
\[
\int_0^{16} \int_{\sqrt{x}}^4\sin(y^3)dydx
\]
Solution: The region
\[
\mathcal{R} = \{(x, y)|0 \leq x \leq 16, \sqrt{x} \leq y \leq 4\}
\]
is given to us in the form of an \( y \)-simple region. Since it is difficult to integrate \( \sin(y^3) \), we are going to rewrite \( \mathcal{R} \) in the form of an \( x \)-simple region. To do this, observe that \( \sqrt{x} \leq y \leq 4 \) implies that \( x \leq y^2 \) and that for any \( y \) in the interval \([0, 4]\), \( x \) has to satisfy only \( 0 \leq x \leq y^2 \). Hence
\[
\mathcal{R} = \{(x, y)|0 \leq y \leq 4, 0 \leq x \leq y^2\}.
\]
Therefore
\[ \int_0^{16} \int_0^{\sqrt{x}} \sin(y^3)dydx = \int_0^{4} \int_0^{y^2} \sin(y^3)dxdy. \]

We now compute
\[
\int_0^{4} \int_0^{y^2} \sin(y^3)dxdy = \int_0^{4} y^2 \sin(y^3)dy
= \int_0^{64} \frac{1}{3} \sin(u)du
= \frac{1}{3} (-\cos(64) - \cos(0)) = \frac{1}{3} (1 - \cos(64)).
\]

\[ \square \]

8. \textit{(10 points)} Compute the volume of the solid in the first octant bounded by 
\[ 2x + 2y + z = 4. \]

\textit{Solution:} We are integrating the function \( f(x,y) = 4 - 2x - 2y \) over the region 
\[ \mathcal{R} = \{(x,y) : x \geq 0, y \geq 0, 2x + 2y \leq 4\} = \{(x,y) : 0 \leq x \leq 2, 0 \leq y \leq 2 - x\}. \]

The volume is thus
\[
\int_0^{2} \int_0^{2-x} (4-2x-2y)dydx = \int_0^{2} (4y - 2xy - y^2)\bigg|_0^{2-x} dx
= \int_0^{2} (x^2 - 4x + 4)dx = (x^3/3 - 2x^2 + 4x)\bigg|_0^{2} = 8/3.
\]

\[ \square \]