MODULI SPACES OF HYPERELLIPTIC CURVES WITH A AND D SINGULARITIES

MAKSYM FEDORCHUK

Abstract. We introduce moduli spaces of quasi-admissible hyperelliptic covers with at worst A and D singularities. Stability conditions for these moduli problems depend on two parameters describing allowable singularities. At the extreme values of the parameters, we obtain the stacks $\mathcal{T}_{A_n}$ and $\mathcal{T}_{D_n}$ of stable limits of A_n and D_n singularities, as well as the quotients of the miniversal deformation spaces of these singularities by a natural $\mathbb{G}_m$-action. We prove that the intermediate spaces are log canonical models of $\mathcal{T}_{A_n}$ and $\mathcal{T}_{D_n}$.

Contents

1. Introduction

We begin a systematic study of the interplay between the local geometry of the miniversal deformation space of a curve singularity and the global geometry of the so-called stack of its stable limits. In this work, we address the case of simple planar curve singularities of types A and D. This choice is explained by the possibility to treat these singularities in a unified fashion using our theory of quasi-admissible hyperelliptic covers. Our main result is the construction of compact moduli spaces of hyperelliptic curves with at worst A and D singularities (Main Theorems 1 and 2). Local structure of these spaces reflects the geometry of miniversal deformation spaces of AD singularities. As a result, we obtain: (1) a complete description of all stable limits of A and D singularities, (2) a generalization of stable reduction, which we call (A_k, D_ℓ)-*stable reduction*, that allows to lower indices of AD singularities appearing in a higher-dimensional family of curves, (3) a simple proof of

2000 *Mathematics Subject Classification.* Primary 14D23; Secondary 14H20, 14B07.

Key words and phrases. Moduli stacks, moduli of curves, singularities.

adjacencies for D singularities. Applications to the log minimal model program for $\overline{M}_g$ are described below.

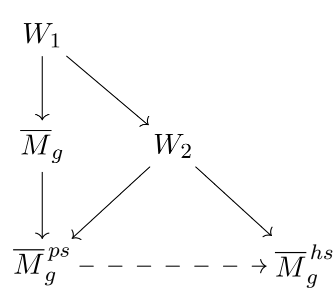
Varieties of stable limits. Given a proper connected but singular curve of arithmetic genus g, its *variety of stable limits* is defined as the totality of all possible stable limits obtained by applying the stable reduction of Deligne and Mumford [DM69] to a smoothing of the curve. We refer the reader to Section 2.1.3 for the precise definition. Understanding varieties of stable limits is crucial to the study of deformation theory of curve singularities on the one hand, and to the study of birational geometry of $\overline{M}_g$ on the other hand.

Applications to the log MMP for $\overline{M}_g$. We now discuss the relevance of our results to the Mori-theoretic study of $\overline{M}_g$ initiated by Hassett and Keel. The ultimate goal of this program is the functorial description of the log canonical models

$$\overline{M}_g(\alpha) := \operatorname{Proj} \bigoplus_{m\geq 0} \mathrm{H}^0(\overline{\mathcal{M}}_g, \lfloor m(K_{\overline{\mathcal{M}}_g} + \alpha\delta)\rfloor).$$

Varieties of stable limits of curve singularities feature prominently in the study of $\overline{M}_g(\alpha)$ because they are very special loci inside $\overline{M}_g$. We list several instances: (1) the variety of stable limits of a genus g curve with a unique A_{2g} ($y^2 = x^{2g+1}$) singularity is the hyperelliptic locus $\overline{H}_g \subset \overline{M}_g$, (2) if $g \equiv 1 \bmod 3$, the variety of stable limits of a genus g curve with a unique $y^3 = x^{g+2}$ singularity is the locus of trigonal curves of the highest Maroni invariant in $\overline{M}_g$; for example the Petri divisor in $\overline{M}_4$ is the variety of stable limits of a curve with a J_{10} ($y^3 = x^6$) singularity.

Because varieties of stable limits often lie in the stable base loci of $K_{\overline{\mathcal{M}}_g} + \alpha\delta$, they appear in factorizations into blow-ups and blow-downs of rational maps between log canonical models of $\overline{M}_g$ that are presently understood due to work of Hassett, Hyeon, and Lee [HH09, HH08, HL10]. For example, consider $\overline{M}_g^{ps} = \overline{M}_g(9/10)$ – the moduli space of at-worst-cuspidal curves, and $\overline{M}_g^{hs} = \overline{M}_g(7/10 - \epsilon)$ – the moduli space of at-worst-tacnodal curves (for precise definitions see [HH09, HH08]). By [HH09], there is a regular morphism $\overline{M}_g \to \overline{M}_g^{ps}$, a divisorial contraction with the exceptional divisor Δ_1 – the locus of curves with *elliptic tails*. By [HH08], there is a rational map $\overline{M}_g^{ps} \dashrightarrow \overline{M}_g^{hs}$, a flip of the locus of curves with *elliptic bridges* to the locus of tacnodal curves. Varieties of stable limits appear as follows: The 1-dimensional fibers of $\overline{M}_g \to \overline{M}_g^{ps}$ are isomorphic to $\mathcal{T}_{A_2}$, the variety of stable limits of the cusp ($y^2 = x^3$); of course $\mathcal{T}_{A_2} \simeq \overline{M}_{1,1}$. Further, there exists a resolution of the rational map $\overline{M}_g^{ps} \dashrightarrow \overline{M}_g^{hs}$:



and the 2-dimensional fibers of $W_1 \to \overline{M}_g^{hs}$ are isomorphic to $\mathcal{T}_{A_3}$, the variety of stable limits of the tacnode ($y^2 = x^4$); in turn $\mathcal{T}_{A_3} \simeq \overline{M}_{1,2}$. The 2-dimensional fibers of $W_2 \to \overline{M}_g^{hs}$ are isomorphic to $\mathcal{H}_3[2]$ of Definition 4.5, which informally can be described as the moduli space of at-worst-cuspidal elliptic bridges.

In a similar fashion, whenever a moduli space $\overline{M}_g[A_k, D_\ell]$ parameterizing proper curves of genus g with at worst A_k and D_ℓ singularities ($k \geq \ell - 1$) is constructed, the variety $\mathcal{T}_{D_\ell} \subset \overline{M}_g$ will appear inside the total transform of the D_ℓ locus under the rational map $\overline{M}_g[A_k, D_\ell] \dashrightarrow \overline{M}_g$.

2. Statement of Main Theorems

In this paper, we introduce and study moduli stacks of (pointed) *quasi-admissible hyperelliptic covers*. We postpone the precise definition to Section 4 (Definitions 4.5 and 4.6) and proceed to describe quasi-admissible covers informally: The quasi-admissible hyperelliptic covers generalize both the *admissible covers* of Harris and Mumford [HM82] and the *twisted covers* of Abramovich, Corti, and Vistoli [ACV03] in the case of degree 2. Namely, a quasi-admissible hyperelliptic cover of genus g is a degree 2 map $\varphi\colon C \to R$ such that

(1) C is a curve of arithmetic genus g,
(2) R is a tree of pointed rational curves, where
(3) the marked points on R are the branch points of φ.

By assigning weights to the branch points, we control the singularities of C: Allowing k branch points to collide introduces an A_{k-1} singularity ($y^2 = x^k$) on C. By forgetting C and φ we obtain a weighted pointed rational curve R. We require R, marked by the weighted branch divisor, to be stable (see Definition 3.1). Finally, in order to have a smooth stack of quasi-admissible covers, we never allow ramification over the nodes of R. As in the theory of twisted covers [ACV03], this is achieved by introducing an orbicurve structure at certain nodes of R and nodes of C lying over them. A new feature of quasi-admissible covers, as compared to admissible and twisted covers, is that C can have singularities over the smooth locus of R. In particular, on the moduli stack there is a boundary divisor δ_{irr} defined as the closure of singular double covers of $\mathbb{P}^1$ (these arise when two branch points come together) and there is a boundary divisor δ_{red} parameterizing covers with a reducible R.

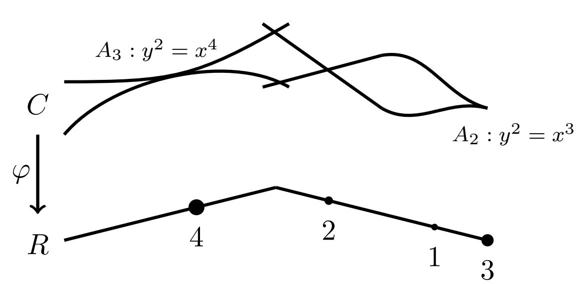


Figure 1. A reducible quasi-admissible hyperelliptic cover of genus 4 with A_3 and A_4 singularities. Numbers indicate multiplicities of the branch divisor.

Finally, a few words on how to obtain quasi-admissible covers with D singularities. For this, we consider a 1-pointed variant of quasi-admissible covers: This is done by introducing a marked point $\chi \in C$. Whenever χ coalesces with an A_k singularity on C, a D_{k+1} singularity appears. The replacement procedure is described in more detail in Section 7, where the equivalence between deformations of a D_{k+1} singularity and an A_k singularity with a section is established. We note that our quasi-admissible covers can have at most one D singularity. The reason for this is that a small deformation of a D singularity has at most one singularity of type D. With the replacement procedure of Section 7 in mind, we say that a quasi-admissible cover has a D_1 (resp., D_2) singularity if χ is a branch point of φ (resp., a node of C lying over a double branch point of φ).

Main Theorem 1 (A_n case)**.** *Let $n \geq 2$ be an integer.*

(1) *For each $k = 1, \dots, n-1$, there exists a smooth and proper Deligne-Mumford stack $\mathcal{H}_n[k]$ representing the functor of quasi-admissible hyperelliptic covers with at worst A_k singularities.*
(2) *The stack $\mathcal{T}_{A_n}$ of stable limits of the A_n singularity is isomorphic to $\mathcal{H}_n[1]$.*
(3) *There is a sequence of divisorial contractions*

$$\mathcal{H}_n[1] \to \mathcal{H}_n[2] \to \dots \to \mathcal{H}_n[n-1].$$

(4) *There is an isomorphism*

$$\mathcal{H}_n[n-1] \simeq \begin{cases} \mathcal{P}(2,3,\dots,n+1), & \text{if } n \text{ is odd}, \\ \mathcal{P}(4,6,\dots,2n+2), & \text{if } n \text{ is even}. \end{cases}$$

(5) *For any $\alpha \in \left(\frac{1}{2}+\frac{1}{k+2}, \frac{1}{2}+\frac{1}{k+1}\right] \cap \mathbb{Q}$, the coarse moduli space of $\mathcal{H}_n[k]$ is*

$$H_n(k) \simeq \operatorname{Proj} R(\mathcal{H}_n[1], K_{\mathcal{H}_n[1]} + \alpha\delta_{irr} + \delta_{red})$$

where δ_{irr}, resp. δ_{red}, is the Cartier divisor of irreducible, resp. reducible, singular covers.

Remark 2.1. Note that the threshold value of α at which $\operatorname{Proj} R(\mathcal{H}_n[1], K_{\mathcal{H}_n[1]}+\alpha\delta_{\mathrm{irr}}+\delta_{\mathrm{red}})$ becomes the coarse moduli space of quasi-admissible covers with at worst A_k singularities is precisely

$$\alpha = \operatorname{lct}(A_k) = \frac{1}{2} + \frac{1}{k+1},$$

where $\operatorname{lct}(A_k)$ is the log canonical threshold of A_k (see Section 2.1.2).

Main Theorem 2 (D_n case)**.** *Let $n \geq 4$ be an integer.*

(1) *For each $1 \leq k \leq n-1$ and $1 \leq \ell \leq \min\{k+1, n-1\}$, there exists a smooth and proper Deligne-Mumford stack $\mathcal{H}_n[k,\ell]$ representing the functor of quasi-admissible 1-pointed hyperelliptic covers with at worst A_k and D_ℓ singularities.*
(2) *The stack $\mathcal{T}_{D_n}$ of stable limits of the D_n singularity is isomorphic to $\mathcal{H}_n[1,2]$.*

(3) *There are divisorial contractions*

$$
\begin{array}{ccccccccc}
\mathcal{H}_n[1,1] & \longrightarrow & \mathcal{H}_n[1,2] & & & & & & \\
\downarrow & & \downarrow & & & & & & \\
\mathcal{H}_n[2,1] & \longrightarrow & \mathcal{H}_n[2,2] & \longrightarrow & \mathcal{H}_n[2,3] & & & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
\cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{H}_n[n-2,1] & \longrightarrow & \mathcal{H}_n[n-2,2] & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \mathcal{H}_n[n-2,n-1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{H}_n[n-1,1] & \longrightarrow & \mathcal{H}_n[n-1,2] & \longrightarrow & \cdots & \longrightarrow & \mathcal{H}_n[n-1,n-2] & \longrightarrow & \mathcal{H}_n[n-1,n-1]
\end{array}
$$

(4) *There is an isomorphism*

$$\mathcal{H}_n[n-1,n-1] = \begin{cases} \mathcal{P}(\frac{n}{2},1,2,3,\dots,n-1), & \textit{if } n \textit{ is even,} \\ \mathcal{P}(n,2,4,6,\dots,2n-2), & \textit{if } n \textit{ is odd.} \end{cases}$$

(5) *For any* $\alpha \in \left(\frac{1}{k+2},\frac{1}{k+1}\right] \cap \mathbb{Q}$ *and* $\beta \in (1-(\ell+1)\alpha, 1-\ell\alpha] \cap \mathbb{Q}$,

$$H_n[k,\ell] \simeq \operatorname{Proj} R(\mathcal{H}_n[1,1], K_{\mathcal{H}_n[1,1]} + (\alpha+1/2)\delta_{irr} + (2\alpha+2\beta-1)\delta_W + \delta_{red}),$$

where δ_{irr} *is the Cartier divisor of irreducible singular covers,* δ_{red} *is the Cartier divisor of reducible singular covers, and* δ_W *is the* Weierstrass divisor *of covers with a marked ramification point.*

We note that A and D singularities come equipped with a $\mathbb{G}_m$-action and so do their versal deformation spaces (see Pinkham [Pin74] for a systematic treatment of singularities with $\mathbb{G}_m$-action). Part (3) of Main Theorem 2 describes $\mathcal{T}_{D_n}$ as an iterated weighted blow-up of a weighted projective space, while Part (4) identifies this weighted projective space with the quotient stack $[\mathrm{Def}(D_n) \smallsetminus \mathbf{0} \ / \ \mathbb{G}_m]$. Part (1) provides a functorial interpretation of the intermediate blow-ups as moduli spaces of quasi-admissible hyperelliptic covers with A and D singularities. (The same applies to Main Theorem 1 if we replace D by A throughout.)

Roadmap of the proof: The moduli stacks $\mathcal{H}_n[k]$ and $\mathcal{H}_n[k,\ell]$ of Main Theorems are defined in Section 4, Definitions 4.5 and 4.6. They are smooth and proper Deligne-Mumford stacks by Theorems 4.9 and 4.10. The morphisms described in Part (3) of Main Theorems are constructed in Section 6. Part (5) of Main Theorems is proved in Section 5 (Theorems 5.5 and 5.7). The proof of Part (4) is contained in Example 4.7. We note that the moduli space $\mathcal{H}_n[n-1]$ is also studied in [ASvdW10], where another proof of Part (4) of Main Theorem 1 (but not Main Theorem 2) can be found. The proof there is deduced from a global quotient construction of $\mathcal{H}_n[n-1]$, valid in arbitrary characteristic, given in [AV04, Theorem 4.1]. Finally, Part (2) of Main Theorems is proved in Section 8.

2.1. **Preliminaries.**

2.1.1. *Curves and their singularities.* Unless specified otherwise, a *curve* is a connected reduced finite type scheme of dimension one over an algebraically closed field. Recall that a curve (singularity) C is *smoothable* if there is a flat family $f\colon \mathcal{C} \to T$ with $f^{-1}(0) \simeq C$ and $f^{-1}(t)$ smooth for $t \neq 0$. All planar curve singularities are smoothable and moreover have nonsingular deformation spaces – these are the only curve singularities encountered in this paper.

A singularity of *type* A_n is analytically isomorphic to $y^2 - x^{n+1} = 0$ at $(0, 0)$. A singularity of *type* D_n is analytically isomorphic to $x(y^2 - x^{n-2}) = 0$ at $(0, 0)$. We note that $A_3 \simeq D_3$.

By a theorem of Arnold (see [Arn75, Arn76, AGLV98]), the A and D singularities, together with the three exceptional singularities E_6, E_7, and E_8, are the only *simple* hypersurface singularities (in every dimension). A singularity is called simple if it admits no nontrivial equisingular deformation, or, equivalently, if it has no *moduli*.

When C is a proper curve, we use $\mathrm{Def}(C)$ to denote the universal deformation space of C, and Δ to denote the *discriminant* – the locus of singular deformations – inside $\mathrm{Def}(C)$. When $p \in C$ is a singular point, we use $\mathrm{Def}(\hat{\mathcal{O}}_{C,p})$ to denote a miniversal deformation space of the singularity. It is a standard fact (see [Ser06, Chapter 3.1] or [Tju69]) that the miniversal deformation space of an isolated hypersurface singularity defined by $f(x_1, \dots, x_n) = 0$ around $(0, \dots, 0)$ is the finite-dimensional $\mathbb{K}$-vector space underlying the *Tjurina algebra*

$$\mathbb{K}[x_1, x_2, \dots, x_n] / \left(f, \partial f/\partial x_1, \dots, \partial f/\partial x_n \right).$$

We record miniversal deformations of A and D singularities together with natural $\mathbb{G}_m$-actions: We have $\mathrm{Def}(A_n) \simeq \mathrm{Spec}\, \mathbb{K}[a_0, \dots, a_{n-1}]$ and the miniversal deformation is given by

$$y^2 - (x^{n+1} + a_{n-1}x^{n-1} + \cdots + a_0) = 0. \tag{2.1}$$

The $\mathbb{G}_m$-action on $\mathrm{Def}(A_n)$ and the miniversal family (2.1) is given by

$$\begin{aligned} &\lambda \cdot (x, y, a_{n-1}, \dots, a_0) = (\lambda^2 x, \lambda^{n+1} y, \lambda^4 a_{n-1}, \dots, \lambda^{2(n+1)} a_0) \quad \text{if } n \text{ is even;} \\ &\lambda \cdot (x, y, a_{n-1}, \dots, a_0) = (\lambda x, \lambda^{\frac{n+1}{2}} y, \lambda^2 a_{n-1}, \dots, \lambda^{n+1} a_0) \quad \text{if } n \text{ is odd.} \end{aligned} \tag{2.2}$$

We have $\mathrm{Def}(D_n) \simeq \mathrm{Spec}\, \mathbb{K}[b, a_0, \dots, a_{n-2}]$ and the miniversal deformation is given by

$$xy^2 + by - (x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0) = 0. \tag{2.3}$$

The $\mathbb{G}_m$-action on $\mathrm{Def}(D_n)$ and the miniversal family (2.3) is given by

$$\begin{aligned} &\lambda \cdot (x, y, b, a_{n-1}, \dots, a_0) = (\lambda x, \lambda^{\frac{n-2}{2}} y, \lambda^{\frac{n}{2}} b, \lambda a_{n-2}, \dots, \lambda^{n-1} a_0) \quad \text{if } n \text{ is even;} \\ &\lambda \cdot (x, y, b, a_{n-1}, \dots, a_0) = (\lambda^2 x, \lambda^{n-2} y, \lambda^n b, \lambda^2 a_{n-2}, \dots, \lambda^{2n-2} a_0) \quad \text{if } n \text{ is odd.} \end{aligned} \tag{2.4}$$

2.1.2. *Log canonical thresholds.* A *log canonical threshold* of a hypersurface quantifies how far the hypersurface is from being a simple normal crossing divisor. We refer the reader to [Kol97, Section 8] for precise definition. Here, we only record the log canonical thresholds of A_n and D_n singularities:

$$\operatorname{lct}(A_n) = \frac{n+3}{2(n+1)} = \frac{1}{2} + \frac{1}{n+1},$$
$$\operatorname{lct}(D_n) = \frac{n}{2(n-1)} = \frac{1}{2} + \frac{1}{2(n-1)}.$$

An amusing fact is that the above thresholds are related to the geometry of the discriminant hypersurface inside the miniversal deformation space of the curve singularity. Namely, let $p \in C \subset \mathbb{K}^2$ be a singularity of type A or D. Let Δ be the discriminant hypersurface inside $\operatorname{Def}(\hat{\mathcal{O}}_{C,p})$. Then

$$\operatorname{lct}(\Delta, \operatorname{Def}(\hat{\mathcal{O}}_{C,p})) = \operatorname{lct}(C, \mathbb{K}^2).$$

2.1.3. *Varieties of stable limits.* We associate to a smoothable curve singularity the variety of all possible stable limits obtained by applying the stable reduction of Deligne and Mumford [DM69] to a smoothing of the singularity. In fact, as follows from the following definition, this variety can be given the structure of a Deligne-Mumford stack.

Definition 2.2 (Stack of stable limits)**.** Let C be a proper integral curve of arithmetic genus g with a single isolated singularity p such that $\hat{\mathcal{O}}_{C,p}$ is smoothable and $\operatorname{Def}(C)$ is irreducible. Consider the rational *moduli map* $j\colon \operatorname{Def}(C) \dashrightarrow \overline{\mathcal{M}}_g$ and its graph

$$\begin{array}{ccc} & Z & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\ \operatorname{Def}(C) & \dashrightarrow & \overline{\mathcal{M}}_g \end{array}$$

We define $\mathcal{T}_{\hat{\mathcal{O}}_{C,p}} := q(p^{-1}(0)) \subset \overline{\mathcal{M}}_g$ to be the *stack of stable limits* of $\hat{\mathcal{O}}_{C,p}$.

The stack of stable limits was introduced by Hassett in [Has00, Section 3], where the description of $\mathcal{T}_{\hat{\mathcal{O}}_{C,p}}$ is obtained for certain toric and quasi-toric planar singularities.

As a variety, $\mathcal{T}_{\hat{\mathcal{O}}_{C,p}}$ is simply the locus of stable curves appearing as stable limits of smoothings of C. If b is the number of analytic branches of $p \in C$ and $\delta(p)$ is the δ-invariant of $\hat{\mathcal{O}}_{C,p}$, then curves in $\mathcal{T}_{\hat{\mathcal{O}}_{C,p}}$ are of the form $\widetilde{C} \cup T$, where $(\widetilde{C}, q_1, \dots, q_b)$ is the pointed normalization of C and $(T, p_1, \dots, p_b)$ is a b-pointed curve of arithmetic genus $\gamma = \delta(p) - b + 1$ ($(T, p_1, \dots, p_b)$ is attached nodally to $(\widetilde{C}, q_1, \dots, q_b)$ by identifying p_i with q_i). The essential information of the stable limit is encoded in $(T, p_1, \dots, p_b)$, called the *tail of a stable limit*. Tails of stable limits are independent of $\widetilde{C}$ and depend only on $\hat{\mathcal{O}}_{C,p}$. It follows that $\mathcal{T}_{\hat{\mathcal{O}}_{C,p}}$ is naturally identified with a closed substack of $\overline{\mathcal{M}}_{\gamma,b}$ (cf. [Has00, Proposition 3.2]).

Much attention is devoted in [Has00] to the case of planar A and D singularities. In particular, Hassett shows that among tails of stable limits of the A_n singularity ($y^2 - x^{n+1} = 0$) occurs every smooth hyperelliptic curve of genus $\lfloor n/2 \rfloor$, marked by a Weierstrass point if n is even, or by two points conjugate under the hyperelliptic involution if n is odd. In the case of the D_n singularity ($x(y^2 - x^{n-2}) = 0$), the picture is similar: Among tails of stable limits occurs every hyperelliptic curve of genus $\lfloor (n-1)/2 \rfloor$, marked by three points – two of which are conjugate – if n is even, or by two points – one of which is a Weierstrass point – if n is odd. This description of tails motivates our definition of quasi-admissible hyperelliptic covers in Section 4.

The following result allows to describe the variety of stable limits of a curve C.

Proposition 2.3. *Suppose that* C *is smoothable and* $\operatorname{Def}(C)$ *is irreducible. Suppose further that there is a proper morphism* $f\colon Y \to \operatorname{Def}(C)$*, with* Y *irreducible, such that* f *is an isomorphism over the locus* $U \subset \operatorname{Def}(C)$ *of smooth deformations of* C *and there exists a family* $\mathcal{X} \to Y$ *of stable curves that agrees with the (uni)versal deformation of* C *over* U*. Then* $\mathcal{T}_C$ *is the image of* $f^{-1}(0)$ *in* $\overline{\mathcal{M}}_g$*, under the natural moduli map* $Y \to \overline{\mathcal{M}}_g$ *induced by the family* $\mathcal{X}$*.*

Proof. Suppose that $g\colon (T,0) \to \operatorname{Def}(C)$ is a smoothing of C. Then by the properness assumption, the map lifts uniquely to $g'\colon (T,0) \to Y$ with $\mathcal{X}_{g'(0)}$ being the stable limit of $T \smallsetminus 0 \to \overline{\mathcal{M}}_g$. Conversely, a smoothing of a stable curve in $f^{-1}(0)$ away from $f^{-1}(0)$ gives a rise to a smoothing of C. □

2.2. Notation and conventions.

2.2.1. *The base field.* We work throughout over an algebraically closed field $\mathbb{K}$ of characteristic 0. The characteristic 0 assumption is used in an essential way at several points of this work. In particular, the stacks $\mathcal{H}_n[k]$ and $\mathcal{H}_n[k,\ell]$ of Main Theorems 1 and 2 are smooth Deligne-Mumford stacks only under the characteristic 0 assumption, as Example 3.5 shows. However, for a fixed integer n, the statements of Main Theorems 1 and 2 remain valid in characteristic p as long as $p > n+1$.

2.2.2. *Stacks and orbicurves.* Ultimately, we are answering a geometric question: Can we give a modular interpretation to a sequence of natural iterated weighted blow-ups of the miniversal deformation spaces of A and D singularities. Not surprisingly, to answer this question we need to use the language of stacks: after all, it is much more fruitful to work with a stack $\overline{\mathcal{M}}_g$ than with its coarse moduli space. Luckily, all of the stacks of interest in this paper are as well-behaved as $\overline{\mathcal{M}}_g$: they parameterize objects with a finite automorphism group and with a smooth deformation space.

An *Artin stack* is a stack that has a separated representable diagonal of finite type (i.e., Isom functors are represented by separated algebraic spaces of finite type) and that admits a representable smooth surjective morphism from a scheme; a *Deligne-Mumford stack* is an Artin stack with an unramified diagonal (i.e., objects have no infinitesimal automorphisms). If $\mathcal{X}$ is a Deligne-Mumford stack, we denote by X its coarse moduli space. An *orbicurve* is a Deligne-Mumford stack of finite type over an algebraically closed field whose coarse moduli space is a curve, and such that the generic stabilizer of every irreducible component is trivial. Unless specified otherwise, a *family* is a flat family of schemes or orbicurves.

2.2.3. *Notation.* The symmetric group on d letters is denoted by $\mathfrak{S}_d$; the cyclic group of order r by μ_r. If D is a $\mathbb{Q}$-Cartier divisor on X, we set

$$R(X,D) := \bigoplus_{m\geq 0} \mathrm{H}^0(X, \lfloor mD \rfloor).$$

The category of schemes of finite type over $\operatorname{Spec} A$ is denoted $\mathfrak{Sch}_A$. If $\mathbb{G}_m$ acts on $\mathbb{K}^{n+1}$ diagonally with weights $a_0,\dots,a_n$, we denote the *weighted projective stack* $\left[\mathbb{K}^{n+1} \smallsetminus \mathbf{0} \,/\, \mathbb{G}_m\right]$ by $\mathcal{P}(a_0,\dots,a_n)$. The coarse moduli space of $\mathcal{P}(a_0,\dots,a_n)$ is $\mathbb{P}(a_0,\dots,a_n)$.

2.3. **Outline of the paper.** In Section 3, we discuss *divisorially marked rational curves*, analogues of weighted pointed curves of [Has03], and the notion of $\mathcal{W}$-stability for them, an analogue of $\mathcal{A}$-stability. The second part of the section deals with *even rational orbicurves* – divisorially marked rational curves endowed with the minimum stack structure allowing for existence of a square root of the marking divisor. In Section 4, we introduce (pointed) quasi-admissible hyperelliptic covers with at worst A (and D) singularities and discuss the notion of $\mathcal{W}$-stability for them. Here, we prove that stacks $\mathcal{H}_n[k]$ and $\mathcal{H}_n[k,\ell]$ of (pointed) quasi-admissible covers are smooth and proper Deligne-Mumford stacks, and show that their local geometry closely reflects the geometry of miniversal deformation spaces of A and D singularities. Section 5 is devoted to the study of certain log canonical divisors on $\mathcal{H}_n[k]$ and $\mathcal{H}_n[k,\ell]$. In particular, we show that the natural reduction morphisms constructed in Section 6 are, on the level of coarse moduli spaces, maps between log canonical models of $\mathcal{T}_{A_n}$ and $\mathcal{T}_{D_n}$. Finally, in Section 8, we list include applications of Main Theorems 1 and 2 to what we call (A,D)-stable reduction as well as discuss the necessity of working with Deligne-Mumford stacks of orbicurves.

Acknowledgments. We would like to thank Jarod Alper, Brendan Hassett, Aise Johan de Jong, and David Smyth for helpful discussions. The whole project was motivated by a question asked by Brendan Hassett to whom we are especially grateful. We thank the organizers of several seminars in which this work was presented before the manuscript was finished. We also thank Sebastian Casalaina-Martin and Radu Laza who shared the preliminary version of [CML10] and encouraged the inclusion of Section 8.

3. Divisorially marked rational curves and even rational orbicurves

3.1. **Divisorially marked rational curves.** We begin by summarizing the theory of pointed and divisorially marked rational curves. Throughout, a *semistable rational curve* will be a proper, connected, at worst nodal curve of arithmetic genus 0. A *divisorially marked rational curve* is a semistable rational curve C together with divisors D_i of degree d_i that are disjoint from $\mathrm{Sing}(C)$. Given $(d_0, d_1, \dots, d_n) \in \mathbb{N}^{n+1}$ and $(w_0, w_1, \dots, w_n) \in \mathbb{Q}^{n+1} \cap (0,1]^{n+1}$, we call the datum $\mathcal{W} := (w_0^{d_0}, w_1^{d_1}, \dots, w_n^{d_n})$ a *weight vector* (we also write w_i^1 as w_i).

Definition 3.1. For a weight vector $\mathcal{W}$, we say that a divisorially marked rational curve is $\mathcal{W}$*-stable* if the following conditions hold:

(1) For every $p \in C$, $\mathrm{mult}_p \sum_{i=0}^{n} w_i D_i \leq 1$.
(2) The line bundle $\omega_C\big(\sum_{i=0}^{n} w_i D_i\big)$ is ample.

Let $\mathcal{R} \to \mathfrak{Sch}_{\mathbb{K}}$ be the *stack of divisorially marked curves.* Its objects are families whose geometric fibers are divisorially marked rational curves.

Definition 3.2. Let $\mathcal{R}_{\mathcal{W}}$ be the full subcategory of $\mathcal{R}$ consisting of families whose geometric fibers are $\mathcal{W}$-stable divisorially marked rational curves. We call $\mathcal{R}_{\mathcal{W}}$ the *stack of rational* $\mathcal{W}$*-stable curves.*

The stack $\mathcal{R}_{\mathcal{W}}$ is closely related to the moduli space of weighted pointed rational curves introduced by Hassett in [Has03]. We recall that for $\mathcal{A} = (a_1, \dots, a_n) \in (0,1]^n \cap \mathbb{Q}^n$, a proper, connected, at worst nodal n-pointed rational curve $(C;p_1,\dots,p_n)$ is $\mathcal{A}$-stable if it satisfies:

(1) p_i are smooth points of C, and if $p_{i_1},\dots,p_{i_k}$ coincide in C, then $\sum_{j=1}^k a_{i_j} \leq 1$.

(2) $\omega_C(\sum\limits_{i=1}^n a_i p_i)$ is ample.

The moduli functor of $\mathcal{A}$-stable curves is represented by a smooth projective scheme $\overline{\mathcal{M}}_{0,\mathcal{A}}$ [Has03, Theorem 2.1]. The relation between $\mathcal{R}_{\mathcal{W}}$ and $\overline{\mathcal{M}}_{0,\mathcal{A}}$ is summarized in the following proposition.

Proposition 3.3. *The category $\mathcal{R}_{\mathcal{W}}$ is a smooth and proper Deligne-Mumford stack over $\mathbb{K}$. Its coarse moduli space $R_{\mathcal{W}}$ is a quotient of $\overline{\mathcal{M}}_{0,\mathcal{A}}$, where* $\mathcal{A} = (\underbrace{w_0,\dots,w_0}_{d_0},\dots,\underbrace{w_n,\dots,w_n}_{d_n})$, *by the action of* $\mathfrak{S} := \mathfrak{S}_{d_0}\times\cdots\times\mathfrak{S}_{d_n}$.

Proof. Consider the forgetful 1-morphism $\mathcal{R}_{\mathcal{W}} \to \mathfrak{M}_0$ to the category of semistable rational curves with at most $1+\sum_{i=0}^n d_i$ nodes. Note that $\mathfrak{M}_0$ is an Artin stack of finite type over $\mathbb{K}$ by, e.g., [Ful05, Proposition 1.10]. Take $T \to \mathfrak{M}_0$ to be a smooth surjective morphism from a separated scheme T of finite type over $\mathbb{K}$. This morphism gives a family $\mathcal{C}_T \to T$ of semistable rational curves. The fiber product $\mathcal{R}_{\mathcal{W}} \times_{\mathfrak{M}_0} T$ is the category whose objects over a T-scheme S are $(n+1)$-tuples of S-flat Cartier divisors $\mathcal{D}_0,\dots,\mathcal{D}_n$ on $\mathcal{C}_T \times_T S$ satisfying the $\mathcal{W}$-stability conditions of Definition 3.1. The morphisms in $\mathcal{R}_{\mathcal{W}} \times_{\mathfrak{M}_0} T$ are obvious Cartesian diagrams and the isomorphisms in $\mathcal{R}_{\mathcal{W}} \times_{\mathfrak{M}_0} T$ are equalities on the nose. It follows that $\mathcal{R}_{\mathcal{W}} \times_{\mathfrak{M}_0} T$ is an open subscheme of $\mathrm{Hilb}_{\mathcal{C}_T/T}$ – the relative Hilbert scheme of $\mathcal{C}_T \to T$. Since $\mathcal{C}_T$ is a separated algebraic space of finite type over T, $\mathrm{Hilb}_{\mathcal{C}_T/T}$ is also an algebraic space of finite type over T by [Ryd10], and so admits an étale surjective morphism from a finite type scheme. Composing this morphism with a smooth surjective morphism $\mathcal{R}_{\mathcal{W}} \times_{\mathfrak{M}_0} T \to \mathcal{R}_{\mathcal{W}}$, we obtain a smooth surjective cover of $\mathcal{R}_{\mathcal{W}}$ by a scheme of finite type over $\mathbb{K}$. Thus, $\mathcal{R}_{\mathcal{W}}$ is an Artin stack of finite type over $\mathbb{K}$.

To show that $\mathcal{R}_{\mathcal{W}}$ is a smooth Deligne-Mumford stack it remains to show that for any $\mathcal{W}$-stable curve defined over a field, infinitesimal automorphisms and obstructions vanish. This follows from the well-known deformation-theoretic result of Proposition 3.4 below.

Since $\mathcal{R}_{\mathcal{W}}$ is a Deligne-Mumford stack of finite type over $\mathbb{K}$, the coarse moduli space exists by [KM97, Corollary 1.3] and the coarse moduli map is proper [Con, Theorem 3.1(1)]. It is easy to see that the coarse moduli space is the scheme-theoretic quotient $\overline{\mathcal{M}}_{0,\mathcal{A}}/\mathfrak{S}$, which is a proper scheme. Thus, $\mathcal{R}_{\mathcal{W}}$ is proper. □

Proposition 3.4. *Let $(C;D_0,\dots,D_n)$ be a $\mathcal{W}$-stable rational curve over $\mathbb{K}$. Then $(C;D_0,\dots,D_n)$ is unobstructed and has no infinitesimal automorphisms.*

Proof. The first statement follows from the fact that C is unobstructed and D_i are Cartier divisors supported at smooth points of C. To prove the second statement, we note that the infinitesimal automorphisms are elements $d \in \mathrm{Hom}(\Omega^1_C,\mathcal{O}_C)$ that satisfy $df \in (f)$ for any local equation f of $D_0+\cdots+D_n$. In characteristic 0, this implies that infinitesimal automorphisms are classified by $\mathrm{Hom}(\Omega^1_C,\mathcal{O}_C(-D)) \simeq \mathrm{H}^0(C,(\omega_C(D))^{-1}) = (0)$. Here, the first isomorphism and the vanishing statement is taken from [Has03, Section 3.3]. □

We remark that by [Has03, Section 3.3.2] the infinitesimal automorphisms of $(C; D_0, \dots, D_n)$ are classified by $\operatorname{Ext}^1(\Omega^1_C\langle D_0, \dots, D_n\rangle, \mathcal{O}_C)$, where $\Omega^1_C\langle D_0, \dots, D_n\rangle$ is the sheaf of differentials on C with logarithmic poles along $D_0, \dots, D_n$.

Example 3.5. A smooth divisorially marked rational curve C over a field $\mathbb{K}$ of positive characteristic may have infinitesimal automorphisms. Indeed, if $\operatorname{char} \mathbb{K} = p$, then by the proof of Proposition 3.4 above, the infinitesimal automorphism $x \mapsto x + \epsilon y$ of $\operatorname{Proj} \mathbb{K}[x, y]$ extends to an automorphism of $\mathbb{P}^1$ divisorially marked by the divisor

$$(x - a_1 y)^{pm_1}(x - a_2 y)^{pm_2} \cdots (x - a_n y)^{pm_n} = 0.$$

3.1.1. *Divisorially marked vs. pointed curves.* Let $\mathcal{A} = \big(\underbrace{w_0, \dots, w_0}_{d_0}, \dots, \underbrace{w_n, \dots, w_n}_{d_n}\big)$ and $\mathcal{W} = \big(w_0^{d_0}, \dots, w_n^{d_n}\big)$. Set $\mathfrak{S} := \mathfrak{S}_{d_0} \times \cdots \times \mathfrak{S}_{d_n}$. We have a sequence of morphisms

$$\overline{\mathcal{M}}_{0,\mathcal{A}} \to \big[\overline{\mathcal{M}}_{0,\mathcal{A}}/\mathfrak{S}\big] \to \mathcal{R}_{\mathcal{W}} \to R_{\mathcal{W}} \simeq \overline{\mathcal{M}}_{0,\mathcal{A}}/\mathfrak{S},$$

defined as follows: The leftmost arrow is the quotient map and the rightmost arrow is the coarse moduli map. To define the middle arrow, let $\mathcal{C}_{\mathcal{A}}$ be the universal family over $\overline{\mathcal{M}}_{0,\mathcal{A}}$. An object of $\big[\overline{\mathcal{M}}_{0,\mathcal{A}}/\mathfrak{S}\big]$ over a scheme T is an $\mathfrak{S}$-torsor $\{P \to T\}$ together with an $\mathfrak{S}$-equivariant morphism $P \to \overline{\mathcal{M}}_{0,\mathcal{A}}$. The middle arrow $\big[\overline{\mathcal{M}}_{0,\mathcal{A}}/\mathfrak{S}\big] \to \mathcal{R}_{\mathcal{W}}$ sends $\{P \to T\}$ to the family $\mathcal{X} \to P/\mathfrak{S} \cong T$ of divisorially marked curves, where $\mathcal{X} \simeq \Big(P \times_{\overline{\mathcal{M}}_{0,\mathcal{A}}} \mathcal{C}_{\mathcal{A}}\Big)/\mathfrak{S}$ and where the divisor D_i on $\Big(P \times_{\overline{\mathcal{M}}_{0,\mathcal{A}}} \mathcal{C}_{\mathcal{A}}\Big)/\mathfrak{S}$ is defined to be the image of weight w_i sections of $P \times_{\overline{\mathcal{M}}_{0,\mathcal{A}}} \mathcal{C}_{\mathcal{A}} \to P$. Note that the middle arrow $\big[\overline{\mathcal{M}}_{0,\mathcal{A}}/\mathfrak{S}\big] \to \mathcal{R}_{\mathcal{W}}$ is not representable since $\mathcal{R}_{\mathcal{W}}$ has smaller stabilizers, as we see in the following example.

Example 3.6. Let $\mathcal{A} = \big(1, 1/3, 1/3, 1/3, 1/3\big)$ and $\mathcal{W} = \big(1, (1/3)^4\big)$. To describe $\overline{\mathcal{M}}_{0,\mathcal{A}}$, note that any $\mathcal{A}$-stable curve has only one irreducible component, namely $\mathbb{P}^1$. Next, we can assume that the point of weight 1 is always at ∞, and 4 points of weight $1/3$ have coordinates x_1, x_2, x_3, x_4 satisfying $x_1 + x_2 + x_3 + x_4 = 0$ on the affine line $\mathbb{P}^1 \smallsetminus \infty$. The stability assumption then translates into $(x_1, x_2, x_3, x_4) \neq (0, 0, 0, 0)$. Since the subgroup of PGL_2 preserving above choices is $\mathbb{G}_m$, we see that $\overline{\mathcal{M}}_{0,\mathcal{A}}$ is the quotient stack

$$[(x_1 + x_2 + x_3 + x_4 = 0) \smallsetminus \mathbf{0} \ / \ \mathbb{G}_m] \simeq \mathbb{P}^2;$$

here, the action of $\mathbb{G}_m$ is given by the usual grading on $S := \mathbb{K}[x_1, x_2, x_3, x_4]$. By the same logic, $\mathcal{R}_{\mathcal{W}}$ can be identified with the quotient stack

$$\Big[\operatorname{Spec}(S/(x_1 + x_2 + x_3 + x_4))^{\mathfrak{S}_4} \smallsetminus \mathbf{0} \ / \ \mathbb{G}_m\Big],$$

where $(S/(x_1 + x_2 + x_3 + x_4))^{\mathfrak{S}_4}$ is the ring of invariants of $S/(x_1 + x_2 + x_3 + x_4)$ under the action of $\mathfrak{S}_4$. The generators of $(S/(x_1 + x_2 + x_3 + x_4))^{\mathfrak{S}_4}$ are elementary symmetric polynomials of degrees 2, 3, and 4. It follows that $\mathcal{R}_{\mathcal{W}}$ is the weighted projective stack $\mathcal{P}(2, 3, 4)$. Finally, $\big[\overline{\mathcal{M}}_{0,\mathcal{A}}/\mathfrak{S}_4\big]$ is $[\operatorname{Proj}(S/(x_1 + x_2 + x_3 + x_4)) \ / \ \mathfrak{S}_4]$, where $\mathfrak{S}_4$ acts by permuting variables. Note that the point with $x_1 = x_2 = x_3 = -x_4/3$ has stabilizer $\mathfrak{S}_3$ in $\big[\overline{\mathcal{M}}_{0,\mathcal{A}}/\mathfrak{S}\big]$ and maps to a point with stabilizer μ_3 in $\mathcal{R}_{\mathcal{W}}$.

3.1.2. *Section at infinity.* In all of the cases under consideration in this paper, the weight vector is of the form $\left(1, w_1^{d_1}, \dots, w_n^{d_n}\right)$. In other words, all families of divisorially marked curves will always carry a distinguished divisor of relative degree 1 and weight 1. Such a divisor defines a section, called *the section at infinity.*

3.1.3. *Odd nodes and odd section at infinity.* If $(C; D_0, D_1, \dots, D_n)$ is a divisorially marked rational curve, where D_0 is the section at infinity, we say that a node of C is *odd* (resp., *even*) if it separates C into two connected components C_1 and C_2 such that D_0 lies on C_1 and such that the degree of $\sum_{i=1}^n D_i$ restricted to C_2 is odd (resp., even). Moreover, we say that the section at infinity is *odd* (resp., *even*) if the total degree $d_1 + \cdots + d_n$ of $\sum_{i=1}^n D_i$ is odd (resp., even). The odd nodes and the odd section at infinity are referred to as *odd points* of C.

3.2. **Even rational orbicurves.** Definitions in this section are inspired by the notion of a twisted cover of [ACV03] (see also [AV02]). We use the gadget of orbicurves, to borrow a metaphor from [ACV03], as a magnifying glass in which apparent singularities of moduli spaces disappear.

Definition 3.7. An *even rational orbicurve* over $\mathbb{K}$ is a triple $(\mathcal{Y}; \tau, D)$, where

(1) $\mathcal{Y}$ is an orbicurve over $\mathbb{K}$ with a coarse moduli space Y.
(2) Y is a semistable rational curve.
(3) $\tau\colon \operatorname{Spec}\mathbb{K} \to \mathcal{Y}$ is a smooth point of Y, called the *section at infinity.*
(4) D is a divisor in the smooth locus of Y, disjoint from $\tau(\operatorname{Spec}\mathbb{K})$.
(5) The points of $\mathcal{Y}$ with non-trivial stabilizers lie exactly over odd points of Y.
(6) Étale locally over an odd node of Y, the orbicurve $\mathcal{Y}$ is isomorphic to
$$\left[\operatorname{Spec}\mathbb{K}[x,y]/(xy)\ /\mu_2\right]$$
where μ_2 acts by $(x,y) \mapsto (-x,-y)$.
(7) Étale locally over an odd section at infinity, the orbicurve $\mathcal{Y}$ is isomorphic to
$$\left[\operatorname{Spec}\mathbb{K}[x]\,/\mu_2\right]$$
where μ_2 acts by $x \mapsto -x$.

The points of $\mathcal{Y}$ with a non-trivial stabilizer are called *odd points.*

Lemma 3.8. *Suppose that $(\mathcal{Y}; \tau, D)$ is an even rational orbicurve over $\mathbb{K}$. Then there is a unique $\mathcal{L} \in \operatorname{Pic}(\mathcal{Y})$ satisfying $\mathcal{L}^2 \simeq \mathcal{O}_{\mathcal{Y}}(D)$.*

Proof. Recall that a line bundle $\mathcal{L}$ on $\mathcal{Y}$ is a datum of a line bundle L on Y together with a character of μ_2 at every odd point of Y. Evidently, for every $\mathcal{L} \in \operatorname{Pic}(\mathcal{Y})$ satisfying $\mathcal{L}^2 \simeq \mathcal{O}_{\mathcal{Y}}(D)$ and for every odd point p of $\mathcal{Y}$, the action of μ_2 on $\mathcal{L} \otimes k(p)$ is given by a non-trivial character. In particular, if $\mathcal{L}^2 \simeq \mathcal{M}^2 \simeq \mathcal{O}_{\mathcal{Y}}(D)$, then $\mathcal{L} \otimes \mathcal{M}^{-1}$ is a pullback of a line bundle from Y. The uniqueness follows because $\operatorname{Pic}(Y)$ is torsion-free and $\operatorname{Pic}(Y) \to \operatorname{Pic}(\mathcal{Y})$ is injective. It remains to establish existence.

We proceed by induction on the number of odd nodes. Suppose there are none. Then either $\mathcal{Y}$ is a scheme and the statement clearly holds, or $\mathcal{Y}$ has an odd section at infinity τ. In the latter case, let $\mathcal{Y}_1$ be the irreducible component of $\mathcal{Y}$ containing τ. Then $\mathcal{Y} = \mathcal{Y}_1 \cup \mathcal{Y}_2$, a nodal union of two connected components. Moreover, $\mathcal{Y}_2$ is a scheme and the degree of D

restricted to every component of $\mathcal{Y}_2$ is even. It follows that there is a line bundle $\mathcal{L}_2$ such that $(\mathcal{L}_2)^2 \simeq \mathcal{O}_{\mathcal{Y}_2}(D|_{\mathcal{Y}_2})$. Set $D_1 := D|_{\mathcal{Y}_1} + p_1$, where p_1 is an arbitrary smooth point of $\mathcal{Y}_1$. Since D_1 has even degree on $\mathcal{Y}_1$, there is a line bundle $\mathcal{L}_1$ such that $(\mathcal{L}_1)^2 \simeq \mathcal{O}_{\mathcal{Y}_1}(D_1)$. Finally, let $\mathcal{L} \in \operatorname{Pic}(\mathcal{Y})$ be such that $\mathcal{L}|_{\mathcal{Y}_1} \simeq \mathcal{L}_1(-\frac{1}{2}\tau)$ and $\mathcal{L}|_{\mathcal{Y}_2} \simeq \mathcal{L}_2$. Clearly, $\mathcal{L}^2 \simeq \mathcal{O}_{\mathcal{Y}}(D)$.

Suppose now $\mathcal{Y}$ has an odd node p. Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be the connected components of $\mathcal{Y}$ such that $\mathcal{Y}_1 \cap \mathcal{Y}_2 = p$ and $\mathcal{Y} = \mathcal{Y}_1 \cup \mathcal{Y}_2$. Take R_1 and R_2 to be irreducible components of $\mathcal{Y}_1$ and $\mathcal{Y}_2$, respectively, containing point p. Let $D_i := D|_{\mathcal{Y}_i} + p_i$, where p_i is an arbitrary smooth point of R_i. Denote by $\widetilde{\mathcal{Y}}_i$ the orbicurve obtained from $\mathcal{Y}_i$ by forgetting the stack structure at p. Then $(\widetilde{\mathcal{Y}}_1, D_1)$ and $(\widetilde{\mathcal{Y}}_2, D_2)$ are even rational orbicurves whose odd nodes are exactly the odd nodes of $(\mathcal{Y}, D)$, with the exception of p. By induction, there are line bundles $\mathcal{L}_i$ on $\widetilde{\mathcal{Y}}_i$ such that $(\mathcal{L}_i)^2 = \mathcal{O}_{\widetilde{\mathcal{Y}}_i}(D_i)$. Now, form a line bundle $\mathcal{L}$ on $\mathcal{Y}$ satisfying $\mathcal{L}|_{\mathcal{Y}_i} = \mathcal{L}_i(-\frac{1}{2}p)$. Then $(\mathcal{L})^2 = \mathcal{O}_{\mathcal{Y}}(D)$.

□

Next, we analyze families of even rational orbicurves over more general bases. As the following lemma illustrates, the presence of the section at infinity greatly simplifies the geometry of an arbitrary divisorially marked family.

Lemma 3.9. *Let $P \to T$ be a $\mathbb{P}^1$-bundle with a section $\tau\colon T \to P$. Suppose that there is a T-flat divisor $D \subset P$ of relative degree d and disjoint from $\tau(T)$. Then there exists a section $\sigma\colon T \to P \smallsetminus \tau(T)$.*

Proof. The idea of the proof is to take the center of mass of the divisor in each fiber (this is where division by d comes in and the characteristic 0 assumption is used). We now formalize this idea. To begin, we show that the section exists affine locally on T. To this end, suppose that T is affine and $P = \mathbb{P}\mathcal{E}$, where $\mathcal{E}$ is a free vector bundle of rank 2. Then $\tau(T)$ is the vanishing locus of some $x \in \mathrm{H}^0(T, \mathcal{E}^*)$, and the relative divisor D is the vanishing locus of $f_D \in \mathrm{H}^0(T, \operatorname{Sym}^d \mathcal{E}^*) = \operatorname{Sym}^d \mathrm{H}^0(T, \mathcal{E}^*)$. Since $\mathcal{E}$ is free of rank 2, we can find another section of $P \to T$ disjoint from $\tau(T)$. It corresponds to $y \in \mathrm{H}^0(T, \mathcal{E}^*)$. We now express f_D in terms of x and y:

$$f_D = a_d x^d + \cdots + a_1 x y^{d-1} + a_0 y^d.$$

Note that the assumption that D is disjoint from $\tau(T)$ implies that $a_0 \neq 0$. We now define $s := y + \dfrac{a_1}{d a_0} x \in \mathrm{H}^0(T, \mathcal{E}^*)$. (Note that when expressed in terms of s and x, the polynomial f_D has no xs^{d-1} term.) The vanishing locus $(s = 0)$ defines a section $\sigma\colon P \to T$, disjoint from $\tau(T)$.

It remains to show that different sections $(s = 0)$ glue. For this, we need to show that the construction of $(s = 0)$ above was in fact independent of the choice of y. Indeed, suppose we chose section $y' = ay + bx$, with $a \neq 0$. Then using $y = (y' - bx)/a$, we rewrite f_D in the new coordinates as

$$\begin{aligned} f_D &= a_d x^d + \cdots + a_1 x((y' - bx)/a)^{d-1} + a_0((y' - bx)/a)^d \\ &= \frac{1}{a^d}(a_d a^d x^d + \cdots + (a a_1 - a_0 d b) x (y')^{d-1} + a_0 (y')^d). \end{aligned}$$

We see that

$$s' = y' + \frac{a a_1 - a_0 d b}{d a_0} x = a y + b x + \frac{a a_1}{d a_0} x - b x = a s.$$

Since $a \neq 0$, s' and s define the same section. □

The principal application of Lemma 3.9 is to the study of the Picard group of a $\mathbb{P}^1$-bundle over a general base. Namely, suppose D is a divisor on the $\mathbb{P}^1$-bundle $P \to T$ of relative degree d, and disjoint from the section at infinity $\tau(T)$. Then by the lemma, there exists a section $\sigma\colon T \to P \smallsetminus \tau(T)$. Denote the image of σ by Σ. The divisor $D - d\Sigma$ is of relative degree 0. Since $\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$, the cohomology and base change theorem implies that $\mathcal{O}_P(D - d\Sigma)$ is a pullback of a line bundle from the base. By the construction, $\mathcal{O}_P(D - d\Sigma)|_{\tau(T)} \simeq \mathcal{O}_T$. It follows that $\mathcal{O}_P(D) \simeq \mathcal{O}_P(d\Sigma)$.

Lemma 3.10. *Let $\pi\colon Y \to T$ be a family of semistable rational curves with the section at infinity $\tau\colon T \to Y$. Suppose that a line bundle $\mathcal{L} \in \mathrm{Pic}(Y)$ has even degree when restricted to every irreducible component of every fiber and satisfies $\tau^*\mathcal{L} \simeq \mathcal{O}_T$. Then there is a unique line bundle $\mathcal{M} \in \mathrm{Pic}(Y)$ satisfying $\mathcal{L} \simeq \mathcal{M}^{\otimes 2}$ and $\tau^*\mathcal{M} \simeq \mathcal{O}_T$.*

Proof. Denote the image of the section at infinity by Θ. The square root of $\mathcal{L}$ restricted to every fiber is unique (by, e.g., Lemma 3.8). Since the fibers are projective, connected, and have $\mathrm{H}^1(X, \mathcal{O}_X) = 0$, by the cohomology and base change theorem there can be at most one line bundle $\mathcal{M}$ such that $\mathcal{M}^{\otimes 2} \simeq \mathcal{L}$ and $\mathcal{M}|_\Theta \simeq \mathcal{O}_\Theta$.

We now prove existence. For every $t \in T$ consider the base extension $\mathcal{O}^{\mathrm{sh}}_{T,t} \to T$ and the pullback family $Y' \to \operatorname{Spec} \mathcal{O}^{\mathrm{sh}}_{T,t}$. On the fiber of Y' over a closed point t, choose smooth points $p_1, \dots, p_d \in Y'_t \smallsetminus \Theta$ such that $\mathcal{L}_t \simeq \mathcal{O}_{Y'_t}(\sum_{i=1}^{d} \varepsilon_i p_i)^{\otimes 2}$, where ε_i are appropriate signs. By smoothness, the points p_i give rise to sections $\sigma_i\colon \operatorname{Spec} \mathcal{O}^{\mathrm{sh}}_{T,t} \to Y'$ with images $\Sigma_i \subset Y'$ [BLR90, 2.2, Prop. 5]. By degree consideration and triviality of $\mathrm{Pic}(\mathcal{O}^{\mathrm{sh}}_{T,t})$, we have $\mathcal{L} \otimes \mathcal{O}_{Y'} \simeq \mathcal{O}_{Y'}(\sum_{i=1}^{d} \varepsilon_i \Sigma_i)^{\otimes 2}$. Moreover, $\mathcal{O}_{Y'}(\sum_{i=1}^{d} \varepsilon_i \Sigma_i)$ restricts to a trivial line bundle along Θ.

It follows that there is a surjective étale cover $T' \to T$ such that the pullback of $\mathcal{L}$ to $Y' := Y \times_T T'$ has a square root $\mathcal{M}$ that restricts to a trivial line bundle along Θ. (By abuse of notation, we denote by Θ the preimage of Θ under any base extension.) Fix an isomorphism $\iota\colon \mathcal{M}_{|\Theta} \to \mathcal{O}_\Theta$. By the uniqueness above, the line bundles $\mathrm{pr}_1^* \mathcal{M}$ and $\mathrm{pr}_2^* \mathcal{M}$ are isomorphic on $Y' \times_Y Y'$. Fix an isomorphism $\alpha\colon \mathrm{pr}_1^* \mathcal{M} \simeq \mathrm{pr}_2^* \mathcal{M}$ such that the following diagram commutes:

$$\begin{array}{ccc}
(\mathrm{pr}_1^* \mathcal{M})|_\Theta & \xrightarrow{\alpha|\Theta} & (\mathrm{pr}_2^* \mathcal{M})|_\Theta \\
\downarrow{\scriptstyle \mathrm{pr}_1^* \iota} & & \downarrow{\scriptstyle \mathrm{pr}_2^* \iota} \\
\mathrm{pr}_1^* \mathcal{O}_\Theta & \xrightarrow{\mathrm{can}} & \mathrm{pr}_2^* \mathcal{O}_\Theta
\end{array}$$

It follows that the cocycle condition $\mathrm{pr}_{12}^*(\alpha) \circ \mathrm{pr}_{23}^*(\alpha) = \mathrm{pr}_{13}^*(\alpha)$ on $Y' \times_Y Y' \times_Y Y'$ is satisfied. Thus, $\mathcal{M}$ descends to a line bundle on Y whose square is $\mathcal{L}$. □

3.3. **Moduli stack of even rational orbicurves.** Fix a weight vector $\mathcal{W} = (1, w_1^{d_1}, \dots, w_n^{d_n})$. A family of *even rational orbicurves* over a scheme T will be a datum $(\pi\colon \mathcal{Y} \to T; D_0, D_1, \dots, D_n)$ where $\pi\colon \mathcal{Y} \to T$ is a flat and proper morphism from a Deligne-Mumford stack such that geometric fibers of π are even rational orbicurves. The divisor D_0 of weight 1 defines the section at infinity $\tau\colon T \to \mathcal{Y}$. From now on, we will conflate D_0 and τ. A family of even rational orbicurves is called $\mathcal{W}$-stable if the coarse moduli space $Y \to T$ is $\mathcal{W}$-stable in the sense of Definition 3.1.

Definition 3.11. We denote by $\mathcal{R}_{\mathcal{W}}^{\text{even}}$ the category of $\mathcal{W}$-stable families of even rational orbicurves. The morphisms in $\mathcal{R}_{\mathcal{W}}^{\text{even}}$ are obvious cartesian diagrams.

In the remainder of this section, we show that the category $\mathcal{R}_{\mathcal{W}}^{\text{even}}$ is a smooth and proper Deligne-Mumford stack over $\mathbb{K}$.

To begin, we recall that given a triple (X, D, r) consisting of an arbitrary Deligne-Mumford stack X, a Cartier divisor D, and a positive integer r, there exists a *root stack* $X_{D,r}$: The objects in $X_{D,r}$ over a scheme T are morphisms $f\colon T \to X$ together with a datum of a triple $(\mathcal{L}, s, \iota)$ consisting of a line bundle $\mathcal{L} \in \operatorname{Pic}(T)$, a section $s \in \mathrm{H}^0(T, \mathcal{L})$ and an isomorphism $\iota\colon \mathcal{L}^r \to f^*\mathcal{O}_X(D)$ satisfying $\iota(s^r) = f^*(D)$. We refer to [Cad07] for the construction of $X_{D,r}$ and the proof that $X_{D,r}$ is Deligne-Mumford.

Next, given a Cartier divisor D with irreducible components $D_1, \ldots, D_k$, we define (with a certain abuse of notation)

$$X\sqrt[r]{D} := X_{D_1,r} \times_X \cdots \times_X X_{D_k,r}.$$

Lemma 3.12. *If X is a proper Deligne-Mumford stack, then so is $X\sqrt[r]{D}$. Moreover, if X is smooth and D is a simple normal crossing divisor with each D_i smooth, then $X\sqrt[r]{D}$ is also smooth.*

Proof. Briefly, étale locally on X, the morphism $X\sqrt[r]{D} \to X$ is given by

$$\Big[\operatorname{Spec} A[x_1, \ldots, x_k]/(x_1^r - f_1, \ldots, x_k^r - f_k) \;/\; (\mu_r)^k\Big] \to \operatorname{Spec} A,$$

where $f_i \in A$ is a local equation of D_i. Our assumptions imply that

$$\operatorname{Spec} A[x_1, \ldots, x_k]/(x_1^r - f_1, \ldots, x_k^r - f_k)$$

is smooth by the Jacobian criterion. Thus $X\sqrt[r]{D}$ is smooth.

To check properness, we use the valuative criterion [LMB00, Proposition 7.12]. Let R be a discrete valuation ring with the uniformizer t and the fraction field K. Consider a morphism $\operatorname{Spec} K \to X_{D_i,r}$. Compose with $X_{D_i,r} \to X$ and use properness of X to conclude that, possibly after a finite base change, there is a unique extension $\phi\colon \operatorname{Spec} R \to X$. It remains to note that since R is a unique factorization domain, we have $\operatorname{Pic}(\operatorname{Spec} R) = 0$ and the r^{th} root of $\phi^*(D_i) \in R$ exists after further base change $t = s^r$, and is unique up to a unit in $R[s]$. Thus, we obtain $\operatorname{Spec} R[s] \to X_{D_i,r}$.

To see that $X\sqrt[r]{D}$ is proper we could also observe that, étale locally, $X\sqrt[r]{D} \to X$ is a map from a stack to its coarse moduli space. In particular, it is proper [Con, Theorem 3.1(1)]. □

Theorem 3.13. *Let $\mathcal{W} = \left(1, w_1^{d_1}, \ldots, w_n^{d_n}\right)$ be a weight vector with $\sum_{i=1}^n d_i$ even. Then $\mathcal{R}_{\mathcal{W}}^{\text{even}}$ is the root stack $\mathcal{R}_{\mathcal{W}}\sqrt[2]{\delta_{\text{odd}}}$, where δ_{odd} is the Cartier divisor of $\mathcal{W}$-stable rational curves with odd nodes. In particular, $\mathcal{R}_{\mathcal{W}}^{\text{even}}$ is a smooth and proper Deligne-Mumford stack over $\mathbb{K}$.*

Proof. Recall that $\mathcal{R}_{\mathcal{W}}$ is a smooth and proper Deligne-Mumford stack by Proposition 3.3. It carries a simple normal crossing Cartier divisor δ_{odd} parameterizing curves with odd nodes. We proceed to establish the equivalence of categories $\mathcal{R}^{\mathrm{even}}_{\mathcal{W}}$ and $\mathcal{R}_{\mathcal{W}}\sqrt[2]{\delta_{\mathrm{odd}}}$.

Consider a family of even rational orbicurves $(\pi\colon \mathcal{Y}\to T; \tau, D_1,\dots,D_n)$. The coarse moduli space Y is a flat and proper family of divisorially marked rational curves over T. Therefore, it induces a morphism $T\to\mathcal{R}_{\mathcal{W}}$. Now suppose $t\in T$ is a point such that the fiber Y_t has k odd nodes $p_1,\dots,p_k$. Denote by δ_i the locus where the node p_i is preserved; δ_i is an irreducible component of δ_{odd}. Note that, since components of δ_{odd} do not self-intersect, the Cartier divisors δ_i are distinct. By definition, the local equation of $\mathcal{Y}\to T$ around p_i is

$$\left[\operatorname{Spec}\mathcal{O}^{\mathrm{sh}}_{T,t}\{x,y\}/(xy-t_i^{a_i})\ /\ \mu_2\right]\to\operatorname{Spec}\mathcal{O}^{\mathrm{sh}}_{T,t},$$

where the action is $(x,y)\mapsto(-x,-y)$, and where $t_i\in\mathcal{O}^{\mathrm{sh}}_{T,t}$ is (a pullback of) the local equation of δ_i. Since

$$\left(\mathcal{O}^{\mathrm{sh}}_{T,t}\{x,y\}/(xy-t_i^{a_i})\right)^{\mu_2}=\mathcal{O}^{\mathrm{sh}}_{T,t}\{x^2,y^2,xy\}/(xy-t_i^{a_i})=\mathcal{O}^{\mathrm{sh}}_{T,t}\{x^2,y^2\}/(x^2y^2-t_i^{2a_i}),$$

the local equation of $Y\to T$ around p_i is

$$\operatorname{Spec}\mathcal{O}^{\mathrm{sh}}_{T,t}\{x^2,y^2\}/(x^2y^2-t_i^{2a_i})\to\operatorname{Spec}\mathcal{O}^{\mathrm{sh}}_{T,t}.$$

Therefore, the morphism $T\to\mathcal{R}_{\mathcal{W}}$ factors étale locally through the root stack $(\mathcal{R}_{\mathcal{W}})_{\delta_i,2}$ for each $i=1,\dots,k$. Thus it factors étale locally through $\mathcal{R}_{\mathcal{W}}\sqrt[2]{\delta_{\mathrm{odd}}}$. Since the étale descent for objects in $\mathcal{R}_{\mathcal{W}}\sqrt[2]{\delta_{\mathrm{odd}}}$ is effective, we obtain a morphism $T\to\mathcal{R}_{\mathcal{W}}\sqrt[2]{\delta_{\mathrm{odd}}}$. Since morphisms in both categories are given by fiber products and since the formation of coarse moduli space commutes with an arbitrary base change in characteristic 0 (e.g., by [AV02, Lemma 2.3.3]), we obtain a natural transformation $\mathcal{R}^{\mathrm{even}}_{\mathcal{W}}\to\mathcal{R}_{\mathcal{W}}\sqrt[2]{\delta_{\mathrm{odd}}}$.

In the other direction, consider a smooth surjective morphism from a scheme T to $\mathcal{R}_{\mathcal{W}}\sqrt[2]{\delta_{\mathrm{odd}}}$. Since $\mathcal{R}_{\mathcal{W}}\sqrt[2]{\delta_{\mathrm{odd}}}$ is smooth and irreducible, T is smooth and can be chosen to be irreducible. The map $T\to\mathcal{R}_{\mathcal{W}}\sqrt[2]{\delta_{\mathrm{odd}}}$ defines a family $(Y\to T;\tau,D_1,\dots,D_n)$ of $\mathcal{W}$-stable rational curves together with a datum $\{(\mathcal{L}_i,t_i): t_i\in\mathrm{H}^0(T,\mathcal{L}_i)\}$, where $s_i=t_i^2$ is a local equation for every irreducible component δ_i of δ_{odd}. The local equation of Y around a node $p_i\in Y_t$ corresponding to δ_i is then

$$\operatorname{Spec}\mathcal{O}^{\mathrm{sh}}_{T,t}\{x,y\}/(xy-s_i^{a_i})=\operatorname{Spec}\mathcal{O}^{\mathrm{sh}}_{T,t}\{x,y\}/(xy-t_i^{2a_i}).$$

Next, we consider the blow-up along the ideal $(x,y,t_i^{a_i})$ (it is a weighted blow-up with weights of x,y,t being $a_i,a_i,1$) and denote by E_i the exceptional divisor of the blow-up. (Note that a_iE_i is Cartier.) The result is a semistable family $Y'\to T$ of rational nodal curves. In the fiber Y_t an odd node p_i of Y_t has been replaced by a rational curve $(E_i)_t$. Evidently, the degree of $\mathcal{O}_{Y'}(aE_i)$ restricted to $(E_i)_t$ is (-2). It follows by Lemma 3.10 that the divisor

$$B:=\sum_{i=1}^n D_i+\sum_{i:\delta_i\subset\delta_{\mathrm{odd}}}E_i$$

is divisible by 2 in the Picard group of Y'. Let $X'\to Y'$ be the μ_2-cover totally branched over B. Then $X'\to T$ is again a family of semistable curves, and we denote by $X\to T$ its stabilization. The action of μ_2 descends to X and the stack $\mathcal{Y}=[X/\mu_2]$ is a family of even rational orbicurves over T. Clearly, the coarse moduli space of $\mathcal{Y}$ is Y. We conclude that there is a morphism $T \to \mathcal{R}_{\mathcal{W}}^{\mathrm{even}}$. Moreover, since our construction commutes with a smooth base change, this morphism descends to give a natural transformation $\mathcal{R}_{\mathcal{W}} \sqrt[2]{\delta_{\mathrm{odd}}} \to \mathcal{R}_{\mathcal{W}}^{\mathrm{even}}$.

Evidently, the two constructed natural transformations define an equivalence of categories. ☐

Corollary 3.14. *The category $\mathcal{R}_{\mathcal{W}}^{\mathrm{even}}$ of even rational orbicurves is a Deligne-Mumford stack over $\mathbb{K}$.*

Proof. In the case $\sum_{i=1}^n d_i$ is even, this is the content of Theorem 3.13 above. The case of odd $\sum_{i=1}^n d_i$ reduces to it by a simple trick: Take $\mathcal{W}' = (1, 1^3, w_1^{d_1}, \dots, w_n^{d_n})$ and consider the morphism $\mathcal{R}_{\mathcal{W}}^{\mathrm{even}} \to \mathcal{R}_{\mathcal{W}'}^{\mathrm{even}}$ that takes a family of even $\mathcal{W}$-stable rational orbicurves and attaches a fixed 4-pointed $\mathbb{P}^1$ to the section at infinity to obtain a $\mathcal{W}'$-stable curve. Clearly, this morphism is a closed immersion and we are done. ☐

4. Quasi-admissible hyperelliptic covers

In this section, we generalize admissible covers of [HM82] twisted covers of [ACV03] in the special case of degree 2. We call the result a *quasi-admissible hyperelliptic cover*. A hyperelliptic quasi-admissible cover differs from a twisted cover in that it is allowed to have non-trivial ramification over a smooth locus and it differs from an admissible cover in that branch points can come together. In particular, it is clear what happens when several branch points collide: a singularity of type A appears on the cover.

In what follows, we define the moduli stack of (pointed) quasi-admissible covers described in the previous paragraph. To motivate this definition, we list the desired specifications of this stack: The stack has to be proper, singularities of its objects need to be controllable, the deformation theory has to be tractable. Our moduli stack of quasi-admissible covers meets all of these requirements: By assigning weights to branch points, we can specify how many can collide and thus specify the allowable singularities of the cover. When more than allowed branch points come together, we let the cover and the target to sprout out additional components. This makes our moduli stack proper. Finally, the deformations of all objects are unobstructed, thus the moduli stack is actually smooth.

Definition 4.1 (Quasi-admissible covers)**.** Define $\mathcal{H} \to \mathfrak{Sch}_{\mathbb{K}}$ to be the stack whose objects over a scheme T are the diagrams

$$\begin{array}{ccccc} \mathcal{X} & \xrightarrow{\varphi} & \mathcal{Y} & \hookleftarrow & D \\ & {\scriptstyle (\tau_1,\tau_2)} \nwarrow & \pi \downarrow \uparrow \tau & & \\ & & T & & \end{array} \tag{4.1}$$

satisfying the following properties:

1. (1) $\pi\colon (\mathcal{Y}; \tau, D) \to T$ is a proper flat family of even rational orbicurves.
2. (2) φ is a finite locally free morphism of degree 2, branched exactly over *the branch divisor* D and étale elsewhere.
3. (3) If τ is odd, that is, locally étale on T we have $\tau\colon\ \operatorname{Spec} A \to [\operatorname{Spec} A[x] \ /\mu_2]$, then φ looks like an étale cover
$$\left[\operatorname{Spec} A[x,t]/(t^2 - u) \ /\mu_2\right] \to [\operatorname{Spec} A[x] \ /\mu_2]\,,$$
where $u \in A^\times$ and the action is $(x,t) \to (-x,-t)$.

(4) If τ is even, then there are sections $\tau_1, \tau_2 \colon T \to \mathcal{X}$ satisfying $\varphi^*(\tau) = \tau_1 + \tau_2$.
(5) Over an odd node $[\operatorname{Spec} A[x,y]/(xy)\ /\mu_2]$ of $\mathcal{Y}$ the morphism φ looks like an étale cover

$$\left[\operatorname{Spec} A[x,y,t]/(xy, t^2-u)\ /\mu_2\right] \to [\operatorname{Spec} A[x,y]/(xy)\ /\mu_2],$$

where $u \in A^\times$ and the action is $(x,y,t) \to (-x,-y,-t)$.

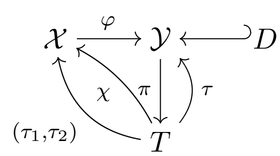
The shorthand notation for an object in $\mathcal{H}$ is $(\varphi\colon \mathcal{X} \to \mathcal{Y}; \tau, D)$. We call $\mathcal{H}$ the *stack of quasi-admissible covers.*

If $\varphi\colon \mathcal{X} \to \mathcal{Y}$ is a quasi-admissible cover over a field $\mathbb{K}$, then passing to the morphism between coarse moduli spaces $\phi\colon X \to Y$, we see that over the odd section at infinity ϕ looks like $\operatorname{Spec} \mathbb{K}[x^2, xt, t^2]/(t^2-u) \to \operatorname{Spec} \mathbb{K}[x^2]$, which becomes $\operatorname{Spec} \mathbb{K}[X, Y^2 - uX] \to \operatorname{Spec} \mathbb{K}[X]$ after the substitution $X = x^2, Y = xt$. Thus, ϕ is ramified over the odd section at infinity. When this happens, we set $\tau_1 := \phi^{-1}(\tau)$. Evidently, τ_1 defines a section of $X \to T$.

By realizing $\mathcal{X}$ as a locally principal subscheme of an $\mathbb{A}^1$-bundle over $\mathcal{Y}$, we see that the only singularities of $\mathcal{X}$ lying over smooth points of $\mathcal{Y}$ are of type A $(y^2 = x^k,\ k \geq 2)$.

Definition 4.2 (Pointed quasi-admissible covers)**.** Let $\mathcal{H}^\chi \to \mathfrak{Sch}_{\mathbb{K}}$ be the stack whose objects over a scheme T are diagrams

(4.2)



such that the diagram obtained by forgetting χ satisfies Definition 4.1. The shorthand notation for an object in $\mathcal{H}^\chi$ is $(\varphi\colon \mathcal{X} \to \mathcal{Y}; \tau, \chi, D)$. We call $\mathcal{H}^\chi$ the *stack of pointed quasi-admissible covers.*

Lemma 4.3. *In Definition 4.1, we have $\mathcal{X} \simeq Spec_{\mathcal{Y}}(\mathcal{O}_{\mathcal{Y}} \oplus \mathcal{L}^{-1})$, where $\mathcal{L}$ is a line bundle satisfying $\mathcal{L}^2 = \mathcal{O}_{\mathcal{Y}}(D)$ and $\tau^*\mathcal{L} \simeq \mathcal{O}_T$. The $\mathcal{O}_{\mathcal{Y}}$-algebra structure on $\mathcal{O}_{\mathcal{Y}} \oplus \mathcal{L}^{-1}$ is given by $\mathcal{L}^{-2} \xrightarrow{\cdot D} \mathcal{O}_{\mathcal{Y}}$.*

Proof. By the characteristic 0 assumption, the trace morphism $\operatorname{Tr}\colon \varphi_*\mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{Y}}$ followed by scaling by $1/2$ gives a splitting $\varphi_*\mathcal{O}_X \simeq \mathcal{O}_{\mathcal{Y}} \oplus \mathcal{L}^{-1}$ with $\mathcal{L}$ a line bundle on $\mathcal{Y}$ satisfying $\mathcal{L}^2 \simeq \mathcal{O}_{\mathcal{Y}}(D)$. Since D is disjoint from τ, twisting by $\pi^*(\tau^*\mathcal{L}^{-1})$ we arrange for $\tau^*\mathcal{L} \simeq \mathcal{O}_T$ without violating the condition that $\mathcal{L}^2 \simeq \mathcal{O}_{\mathcal{Y}}(D)$. Finally, $\mathcal{X} \simeq Spec_{\mathcal{Y}}(\mathcal{O}_{\mathcal{Y}} \oplus \mathcal{L}^{-1})$ because φ is a finite morphism. □

Definition 4.4 (Branch morphisms)**.** The *branch morphism* $\mathbf{br}\colon \mathcal{H} \to \mathcal{R}$ is a forgetful functor sending the family in Display (4.1) to the family of divisorially marked rational curves $(Y; \tau, D)$. The *branch morphism* $\mathbf{br}\colon \mathcal{H}^\chi \to \mathcal{R}$ is a forgetful functor sending the family in Display (4.2) to the family of divisorially marked rational curves $(Y; \tau, \varphi(\chi(T)), D)$.

Recall from Section 3 that $\mathcal{R}_{\mathcal{W}}$ is the moduli stack of $\mathcal{W}$-stable divisorially marked rational curves. Below we use the branch morphism to define *stability conditions* for quasi-admissible covers with A and D singularities.

Definition 4.5 (Stable quasi-admissible covers with A singularities)**.** Fix an integer $n \geq 2$ and a rational number $\alpha \in (0, 1/2]$. Consider the weight vector $\mathcal{W} = \big(1, \alpha^{n+1}\big)$. We define $\mathcal{H}_{n,\alpha}$ to be the moduli stack of quasi-admissible covers $(\varphi\colon \mathcal{X} \to \mathcal{Y}; \tau, D)$ such that $(\mathcal{Y}; \tau, D)$ is a $\mathcal{W}$-stable curves. Alternatively, $\mathcal{H}_{n,\alpha} = \mathcal{H} \times_{\mathcal{R}} \mathcal{R}_{\mathcal{W}}$, where $\mathcal{H}$ maps to $\mathcal{R}$ via the branch morphism of Definition 4.4. The objects of $\mathcal{H}_{n,\alpha}$ are called *$\mathcal{W}$-stable quasi-admissible covers*.

If $1/(k+2) < \alpha \leq 1/(k+1)$ for some $k \in \{1, \dots, n-1\}$, then we denote $\mathcal{H}_{n,\alpha}$ by $\mathcal{H}_n[k]$. For such α the geometric points of $\mathcal{H}_{n,\alpha} = \mathcal{H}_n[k]$ are quasi-admissible covers $(\varphi\colon \mathcal{X} \to \mathcal{Y}; \tau, D)$ such that

(1) the branch divisor D is of degree $n+1$ and of weight α,
(2) the section at infinity τ is of weight 1,
(3) $(Y; \tau, D)$ is a $\mathcal{W}$-stable rational curve,
(4) $\mathcal{X}$ has at worst A_k singularities,
(5) The arithmetic genus of X is $\left\lfloor \dfrac{n}{2} \right\rfloor$.

Definition 4.6 (Stable quasi-admissible covers with D singularities)**.** Fix an integer $n \geq 4$ and rational numbers $\alpha \in (0, 1/2]$ and $\beta \in (0, 1-\alpha]$. Consider the weight vector $\mathcal{W} = (1, \beta, \alpha^n)$. We define the stack $\mathcal{H}_{n,\alpha,\beta}$ to be the fiber product $\mathcal{H}^\chi \times_{\mathcal{R}} \mathcal{R}_{\mathcal{W}}$, where $\mathcal{H}^\chi$ maps to $\mathcal{R}$ via the branch morphism of Definition 4.4. The objects of $\mathcal{H}_{n,\alpha,\beta}$ are called *$\mathcal{W}$-stable pointed quasi-admissible covers*.

We now explicate the meaning of $\mathcal{W}$-stability for pointed quasi-admissible covers. To this end, consider the unique integers k and ℓ such that α and β satisfy inequalities

$$\begin{gathered} \frac{1}{k+2} < \alpha \leq \frac{1}{k+1}, \\ 1 - (\ell+1)\alpha < \beta \leq 1 - \ell\alpha. \end{gathered} \tag{4.3}$$

If $(\varphi\colon \mathcal{X} \to \mathcal{Y}; \tau, \chi, D)$ is a geometric point of $\mathcal{H}_{n,\alpha,\beta}$, then

(1) $\mathcal{X}$ has at worst A_k singularities, since at most $k+1$ branch points can coalesce,
(2) the marked point χ can coalesce with (at worst) $A_{\ell-1}$ singularity on $\mathcal{X}$,
(3) The arithmetic genus of X is $\lfloor (n-1)/2 \rfloor$.

Because of the above, we denote $\mathcal{H}_{n,\alpha,\beta}$ by $\mathcal{H}_n[k,\ell]$ when emphasizing the singularities allowed on X.

Example 4.7 (A case study of $\mathcal{H}_n[n-1]$ and $\mathcal{H}_n[n-1, n-1]$)**.**

We begin our study of quasi-admissible covers by proving Parts (4) of Main Theorems 1 and 2. First, we deal with the case of $\mathcal{H}_n[n-1]$, the moduli stack of $\mathcal{W}$-stable quasi-admissible covers, where $\mathcal{W} = (1, (1/n)^{n+1})$. The stability assumption on $(\varphi\colon \mathcal{X} \to \mathcal{Y}; \tau, D)$ implies that $\mathcal{Y}$ is necessarily $\mathbb{P}^1$ and the support of the branch divisor D has at least 2 distinct points, i.e., not all $n+1$ points can come together. We denote by ∞ the section at infinity. Since it has weight 1, no branch point can lie at ∞. Further analysis depends on the parity of n.

We first treat the case of n odd. In this case, a hyperelliptic cover is uniquely determined by a branch divisor. It remains to describe the moduli space of degree $n+1$ divisors on $\mathbb{A}^1$ not supported at a single point. In characteristic 0 every such divisor can be brought into the *normal form*

$$x^{n+1} + a_{n-1}x^{n-1} + \cdots + a_0 = 0,$$

with not all a_i zero. A subgroup of automorphisms of $\mathbb{P}^1$ fixing ∞ and preserving the normal form is $\mathbb{G}_m$. It acts on $\operatorname{Spec} \mathbb{K}[a_{n-1}, \dots, a_0] \smallsetminus \mathbf{0}$ diagonally with weights $(2, \dots, n+1)$. It follows that our moduli space is the weighted projective stack $\mathcal{P}(2, 3, \dots, n+1)$.

A reader might recall that $\mathcal{P}(2, 3, \dots, n+1)$ is the stack whose objects over a scheme T are data $(\mathcal{L}, s_2, \dots, s_{n+1})$ of a line bundle $\mathcal{L} \in \operatorname{Pic}(T)$ and sections $s_m \in \mathrm{H}^0(T, \mathcal{L}^m)$, $m = 2, \dots, n+1$. We briefly sketch another proof of the equivalence between $\mathcal{H}_n[n-1]$ and $\mathcal{P}(2, 3, \dots, n+1)$:

Given a $\mathcal{W}$-stable cover $(\varphi\colon \mathcal{X} \to \mathcal{Y}; \tau, D)$, where $\mathcal{W} = (1, (1/n)^{n+1})$, all geometric fibers of $\pi\colon \mathcal{Y} \to T$ are isomorphic to $\mathbb{P}^1$. The existence of τ implies that $\pi\colon \mathcal{Y} \to T$ is a $\mathbb{P}^1$-bundle. Since $D \subset \mathcal{Y}$ is a T-flat divisor on $\mathcal{Y}$ disjoint from $\tau(T)$, by Lemma 3.9 there exists a section $\sigma\colon T \to \mathcal{Y}$, also disjoint from $\tau(T)$. Consider the short exact sequence

$$0 \to \mathcal{O}_{\mathcal{Y}}(\sigma(T) - \tau(T)) \to \mathcal{O}_{\mathcal{Y}}(\sigma(T)) \to \mathcal{O}_{\tau(T)}(\sigma(T)) \to 0. \tag{4.4}$$

Pushing (4.4) via π to T, we obtain a split short exact sequence

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{O}_T \to 0,$$

where $\mathcal{L} \simeq \tau^* \mathcal{O}_{\mathcal{Y}}(-\tau(T)) \simeq \sigma^* \mathcal{O}_{\mathcal{Y}}(\sigma(T))$ and $\mathcal{E} = \pi_* \mathcal{O}_{\mathcal{Y}}(\sigma(T))$. It follows that $\mathcal{Y} \simeq \mathbb{P}(\mathcal{E}) \simeq \mathbb{P}(\mathcal{O}_T \oplus \mathcal{L})$. The divisor D is the vanishing locus of a section $s_D \in \mathrm{H}^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}(D))$. Using $D \sim (n+1)\sigma(T)$, we rewrite

$$\mathrm{H}^0(\mathcal{Y}, D) = \mathrm{H}^0(T, \pi_* \mathcal{O}_{\mathcal{Y}}(D)) = \mathrm{H}^0(T, \operatorname{Sym}^{n+1}(\mathcal{O}_T \oplus \mathcal{L})) = \bigoplus_{m=0}^{n+1} \mathrm{H}^0(T, \mathcal{L}^m).$$

It follows that a $\mathcal{W}$-stable cover defines a line bundle $\mathcal{L}$ and sections $s_m \in \mathrm{H}^0(T, \mathcal{L}^m)$ for each $m = 2, \dots, n+1$; note that the section of $\mathcal{L}$ defined by D is actually 0. The construction of the $\mathcal{W}$-stable cover from the datum of a line bundle $\mathcal{L}$ and sections $s_m \in \mathrm{H}^0(T, \mathcal{L}^m)$, $m = 2, \dots, n+1$, is straightforward. It follows that there is an equivalence between the category of $(1, (1/n)^{n+1})$-stable covers and $\mathcal{P}(2, 3, \dots, n+1)$ in the case of odd n.

The case of n even is treated similarly: The only minor difference is that the section at infinity is now odd. Given a $\mathcal{W}$-stable cover $(\varphi\colon \mathcal{X} \to \mathcal{Y}; \tau, D)$, the section $\tau(T)$ lies in the branch locus of the map $\phi\colon X \to Y$ between the coarse moduli spaces. As in the case of odd n, Y is a $\mathbb{P}^1$-bundle and the existence of the divisor $D \subset Y$ gives a section $\sigma\colon T \to Y$ disjoint from $\tau(T)$. We have $\pi_* \mathcal{O}_Y(\sigma(T)) = \mathcal{O}_T \oplus \mathcal{L}$. As before, the divisor D defines sections $s_m \in \mathrm{H}^0(T, \mathcal{L}^m)$ for each $m = 2, \dots, n+1$. It remains to observe that by our construction the line bundle $\mathcal{L} \simeq \tau^* \mathcal{O}_Y(-\tau(T))$ has a square root because $\mathcal{O}_Y(\tau(T))$ pulls back to the line bundle $\mathcal{O}_X(2\tau_1(T))$ via ϕ. Hence, we can write $\mathcal{L} \simeq \mathcal{M}^2$. We conclude that giving a family of $\mathcal{W}$-stable covers over T is equivalent to giving a line bundle $\mathcal{M}$ on T and sections $s_m \in \mathrm{H}^0(T, \mathcal{M}^{2m})$ for each $m = 2, \dots, n+1$. This shows that $\mathcal{H}_n[n-1] \simeq \mathcal{P}(4, 6, \dots, 2(n+1))$.

We remark that in positive characteristic the stack $\mathcal{H}_n[n-1]$ can have nonreduced stabilizers, as Example 3.5 shows. Furthermore, in positive characteristic, $\mathcal{H}_n[n-1]$ is in general not isomorphic to $\mathcal{P}(2, 3, \dots, n+1)$ or $\mathcal{P}(4, 6, \dots, 2(n+1))$. We refer to [AV04] for the description of the stack of degree 2 cyclic covers of $\mathbb{P}^1$ in this case.

Finally, we briefly sketch what happens in the case of $\mathcal{H}_n[n-1, n-1]$. Take $\beta = \alpha = 1/n$, and for $\mathcal{W} = (1, \beta, \alpha^n)$, consider a $\mathcal{W}$-stable pointed quasi-admissible cover $(\varphi\colon \mathcal{X} \to \mathcal{Y}; \tau, \chi, D)$. The coarse moduli space $Y \to T$ is a $\mathbb{P}^1$-bundle with a section

$\sigma := \varphi \circ \chi \colon T \to Y$ disjoint from $\tau(T)$ and such that $D \sim n\sigma(T)$. Moreover, we have $\pi_* \mathcal{O}_Y(\sigma(T)) = \mathcal{O}_T \oplus \mathcal{L}$, where $\mathcal{L} \simeq \tau^* \mathcal{O}_Y(-\tau(T)) \simeq \sigma^* \mathcal{O}_Y(\sigma(T))$. As before, we see that D defines sections $s_m \in \mathrm{H}^0(T, \mathcal{L}^m)$ for $m = 1, \dots, n$ and that $\mathcal{L}$ has a square root when n is odd (but s_1 is not zero because σ is not the center of mass of D). It remains to observe that in the presence of the section $\chi \colon T \to X$, the line bundle $\sigma^* \mathcal{O}_Y(D)$ has a square root because D pulls back via φ to the square of the ramification divisor $\mathrm{Ram}(\varphi)$. Since $\mathcal{L}^n \simeq \sigma^* \mathcal{O}_Y(n\sigma(T)) \simeq \sigma^* \mathcal{O}_Y(D)$, we conclude that $\mathrm{Ram}(\varphi)$ gives us a section of the square root of $\mathcal{L}^n$. It is now clear that

$$\mathcal{H}_n[n-1, n-1] = \begin{cases} \mathcal{P}(\frac{n}{2}, 1, 2, 3, \dots, n-1), & \text{if } n \text{ is even}, \\ \mathcal{P}(n, 2, 4, 6, \dots, 2n-2), & \text{if } n \text{ is odd}. \end{cases}$$

We note that the explicit presentation of $\mathcal{H}_n[n-1]$ given above in the case of odd n has an added advantage of proving that $\mathcal{H}_n[n-1] \simeq [\mathrm{Def}(A_n) \smallsetminus \mathbf{0} \,/\mathbb{G}_m]$, where the $\mathbb{G}_m$ action is given by Equation (2.2). Similar arguments apply to give the following result whose proof we omit.

Corollary 4.8. *For $\mathbb{G}_m$ actions on $\mathrm{Def}(A_n)$ and $\mathrm{Def}(D_n)$ given in Equations (2.2) –(2.4),*
1. $\mathcal{H}_n[n-1] \simeq [\mathrm{Def}(A_n) \smallsetminus \mathbf{0} \,/\mathbb{G}_m]$,
2. $\mathcal{H}_n[n-1, n-1] \simeq [\mathrm{Def}(D_n) \smallsetminus \mathbf{0} \,/\mathbb{G}_m]$.

The following two theorems establish that $\mathcal{H}_n[k]$ and $\mathcal{H}_n[k, \ell]$ are proper and smooth Deligne-Mumford stacks over $\mathbb{K}$.

Theorem 4.9. *The stacks $\mathcal{H}_n[k]$ and $\mathcal{H}_n[k, \ell]$ are proper Deligne-Mumford stacks over $\mathbb{K}$.*

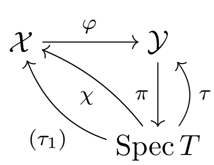
Proof. Evidently, the forgetful natural transformation $\mathcal{H}_n[k, \ell] \to \mathcal{H}_n[k]$ is representable by proper Deligne-Mumford stacks. Thus we reduce to the case of $\mathcal{H}_n[k]$. By Definition 4.1, the branch morphism $\mathbf{br} \colon \mathcal{H}_n[k] \to \mathcal{R}_{\mathcal{W}}$ factors through $\mathcal{R}_{\mathcal{W}}^{\mathrm{even}}$. Given a family of even rational orbicurves, there exists a unique quasi-admissible cover over it (with μ_2 of extra automorphisms if n is even). Thus, the morphism $\mathcal{H}_n[k] \to \mathcal{R}_{\mathcal{W}}^{\mathrm{even}}$ is representable by proper Deligne-Mumford stacks. Finallly, $\mathcal{R}_{\mathcal{W}}^{\mathrm{even}}$ is a proper Deligne-Mumford stack over $\mathbb{K}$ by Theorem 3.13. This finishes the proof. □

4.1. **Deformation theory.**

Theorem 4.10. *$\mathcal{H}_n[k]$ and $\mathcal{H}_n[k, \ell]$ are smooth Deligne-Mumford stacks over $\mathbb{K}$.*

Consider a pointed quasi-admissible cover over $T = \operatorname{Spec} \mathbb{K}$ as in Display (4.2). Note that the divisor D is uniquely determined by φ. Similarly, when τ is even, the section τ_1 determines τ_2. Therefore, we reduce to the study of the deformation theory of

(4.5)



Note that the deformations of the closed immersions $\tau_1 \colon \operatorname{Spec} \mathbb{K} \to \mathcal{X}$, if τ is even, and respectively of $\tau \colon \operatorname{Spec} \mathbb{K} \to \mathcal{Y}$, if τ is odd, are unobstructed because τ lies in $\mathcal{Y}^{\mathrm{smooth}} \smallsetminus D$

by Definition 4.1. It follows that the obstructions to deforming Diagram (4.5) vanish as long as the diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\varphi} & \mathcal{Y} \\ & \nwarrow^{\chi} & \downarrow{\scriptstyle \pi} \\ & & \operatorname{Spec} T \end{array} \tag{4.6}$$

is unobstructed.

Suppose now that $(\varphi\colon \mathcal{X} \to \mathcal{Y}; \chi)$ is a cover as in Diagram (4.6) over $T = \operatorname{Spec} A$, where A is a local Artinian $\mathbb{K}$-algebra. Set $D = \mathbf{br}(\varphi)$. We say that an open affine cover $\mathfrak{U} = \{U_i\}_{i\in S}$ of $\mathcal{Y}$ is *adapted* if every point of $D + \varphi(\chi) + \operatorname{Sing}(\mathcal{Y})$ is contained in exactly one U_i.

Lemma 4.11. *Let $(\varphi\colon \mathcal{X} \to \mathcal{Y}; \chi)$ be a quasi-admissible cover over a local Artinian $\mathbb{K}$-algebra A, and $\mathfrak{U} = \{U_i\}_{i\in S}$ be an adapted affine cover of $\mathcal{Y}$. Consider a $\mathbb{K}$-extension $0 \to I \to A' \to A \to 0$. Then any collection of deformations of $\varphi|_{U_i}$ over A', for every $i \in S$, gives rise to a global deformation of $(\varphi\colon \mathcal{X} \to \mathcal{Y}; \chi)$ over A'.*

Proof. By the assumption, the intersections $U_i \cap U_j$ are smooth affine schemes and so have only trivial deformations. It follows that there is an isomorphism $g_{ij} \in \mathrm{H}^0(U_{ij}, T^1 \otimes I)$ between the restrictions to U_{ij} of deformations over U_i and U_j; here, $T^1 = \mathcal{H}om(\Omega^1_Y, \mathcal{O}_Y)$. The vanishing of the associated 2-cocycle in $\mathrm{H}^2(Y, T^1 \otimes I)$ is a necessary and sufficient condition for the existence of the requisite global extension of $(\varphi\colon \mathcal{X} \to \mathcal{Y}; \chi, D)$ to A'. But $\mathrm{H}^2(Y, T^1 \otimes I) = 0$ because Y is proper of dimension one. □

Proposition 4.12. *Let $\varphi\colon \mathcal{X} \to \mathcal{Y}$ be a quasi-admissible cover over* $\operatorname{Spec} \mathbb{K}$. *For any adapted cover $\mathfrak{U} = \{U_i\}_{i\in S}$ of $\mathcal{Y}$ the map*

$$\operatorname{Def}(\varphi) \to \prod_{i\in S} \operatorname{Def}(\varphi|_{U_i})$$

is formally smooth.

Proof. This is Lemma 4.11. □

Proof of Theorem 4.10: Let $(\varphi\colon \mathcal{X} \to \mathcal{Y}; \tau, \chi, D)$ be a pointed quasi-admissible cover over $\mathbb{K}$. By Proposition 4.12, we need only to check that for every affine chart U in an adapted open cover $\mathfrak{U} = \{U_i\}_{i\in S}$ of $\mathcal{Y}$, the deformation space $\operatorname{Def}(\varphi|U_i)$ is formally smooth. The analysis breaks down into several cases.

Case I: *U contains an (orbi)node.* The module of Kähler differentials of $B = A[x,y]/(x,y)$ has a two-term free resolution $0 \to B \xrightarrow{\binom{y}{x}} B \oplus B \to \Omega^1_{B/A} \to 0$. As observed in [AV02], the above resolution is μ_2-equivariant for the action $(x,y) \to (-x,-y)$. It follows that locally at the (orbi)node the projective dimension of Ω^1_U is 2. In particular, $\operatorname{Ext}^2(\Omega^1_U, \mathcal{O}_U) = 0$ and $\operatorname{Def}(U)$ is smooth. It remains to observe that since φ is finite étale over U, the map $\operatorname{Def}(\varphi|_U) \to \operatorname{Def}(U)$ is formally smooth.

Case II: *U is smooth and contains $\varphi(\chi)$.* Let $U = \operatorname{Spec} A[x]$ and $\varphi^{-1}(U) = \operatorname{Spec} A[x]/(y^2 - f(x))$. Without loss of generality, the section χ is $(x = 0, y = a_0)$, where $a_0^2 = f(0)$. For an arbitrary $\mathbb{K}$-extension $A' \to A \to 0$, we would like to find a lifting of $f(x)$ and a_0 to $A'[x]$ and A', respectively. Write $f(x) = xg(x) + a_0^2$. Take $g'(x)$ to be an arbitrary lifting of $g(x)$

and a_0' be an arbitrary lifting of a_0. Then $f'(x) = xg'(x) + (a_0')^2 \in A'[x]$ is a lifting of $f(x)$ to $A'[x]$ and the section $\chi' : (x = 0, y = a_0')$ is a lifting of χ.
Case III: *U is smooth and does not contain $\varphi(\chi)$.* This is the simplest case: the deformations of $\varphi|_U$ are the same as deformations of U together with the divisor $D \cap U$. The deformation space is smooth because U is a smooth one dimensional affine scheme. □

Theorem 7.1 of Section 7 establishes the equivalence between deformations of an A_{k-1} singularity with a section and a D_k singularity. Using it, we rephrase Proposition 4.12 as follows.

Proposition 4.13. *Let $(\varphi\colon\ \mathcal{X} \to \mathcal{Y}; \tau, \chi, D)$ be a pointed quasi-admissible cover. Suppose that $D = k_0 p_0 + \sum_{i=1}^{r} (k_i + 1) p_i$, where p_i are distinct points of $\mathcal{Y}$. Further assume that $\varphi(\chi) = p_0$. Then the natural morphism*

$$\mathrm{Def}(\varphi) \to \mathrm{Def}(D_{k_0}) \times \mathrm{Def}(A_{k_1}) \times \cdots \times \mathrm{Def}(A_{k_r})$$

is formally smooth.

5. The log minimal model program for $\mathcal{T}_{A_n}$ and $\mathcal{T}_{D_n}$

In this section, we prove Parts 5 of Main Theorems 1 and 2. The key part of the argument is the positivity of certain log canonical divisors on $\mathcal{H}_{n,\alpha}$ and $\mathcal{H}_{n,\alpha,\beta}$, which we establish using [Fed11]. Throughout, we use standard notation for the divisor classes on moduli spaces of pointed curves.

We begin by introducing natural Cartier divisor classes on $\mathcal{H}_{n,\alpha}$ and $\mathcal{H}_{n,\alpha,\beta}$, which we define in terms of divisor classes on moduli spaces of weighted pointed curves. Given a weight vector $\mathcal{A} = \big(1, \underbrace{\alpha, \dots, \alpha}_{n+1}\big)$ or $\mathcal{A} = \big(1, \beta, \underbrace{\alpha, \dots, \alpha}_{n}\big)$, consider the moduli space $\overline{\mathcal{M}}_{0,\mathcal{A}}$ of $\mathcal{A}$-stable rational curves [Has03]. We recall that irreducible boundary divisors on $\overline{\mathcal{M}}_{0,\mathcal{A}}$ are parameterized by partitions $I \cup J = \mathcal{A}$. We arrange I to always contain weight 1. Set

$$\Delta_{\text{odd}} := \sum_{|J| \text{ is odd and } \sum_{i\in J} \alpha_i > 1} \Delta_{I,J}\,, \qquad \Delta_s := \sum_{|J|=2,\ \sum_{i\in J}\alpha_i \le 1} \Delta_{I,J}\,,$$

$$\Delta_{\text{even}} := \sum_{|J| \text{ is even and } \sum_{i\in J} \alpha_i > 1} \Delta_{I,J}\,, \qquad \Delta := \Delta_{\text{odd}} + \Delta_{\text{even}}\,.$$

When $\mathcal{A} = \big(1, \beta, \alpha, \dots, \alpha\big)$, we also denote by $\Delta_{\sigma\cap\chi}$ the divisor of curves $(C; \tau, \chi, \{\sigma_i\}_{i=1}^n)$ in $\overline{\mathcal{M}}_{0,\mathcal{A}}$ such that $\chi = \sigma_i$ for some i. In other words, $\Delta_{\sigma\cap\chi}$ parameterizes curves where the point of weight β coincides with one of the points of weight α. When $\alpha + \beta > 1$, we take $\Delta_{\sigma\cap\chi} = 0$.

Let $\mathbf{q}\colon\ \overline{\mathcal{M}}_{0,\mathcal{A}} \to \overline{\mathcal{M}}_{0,\mathcal{A}}/\mathfrak{S}$ be the quotient map, where $\mathfrak{S} = \mathfrak{S}_{n+1}$ if $\mathcal{A} = (1, \underbrace{\alpha, \dots, \alpha}_{n+1})$, and $\mathfrak{S} = \mathfrak{S}_n$ if $\mathcal{A} = (1, \beta, \underbrace{\alpha, \dots, \alpha}_{n})$. For $\mathcal{W} = (1, \alpha^{n+1})$ or $\mathcal{W} = (1, \beta, \alpha^n)$, we abuse notation and denote $\overline{\mathcal{M}}_{0,\mathcal{A}} \to \mathcal{R}_{\mathcal{W}}$ also by $\mathbf{q}$ (cf. Section 3.1.1). This abuse is inconsequential to the study of divisors, since the coarse moduli map $\mathcal{R}_{\mathcal{W}} \to \overline{\mathcal{M}}_{0,\mathcal{A}}/\mathfrak{S}$ is an isomorphism in codimension one. Given $D \subset \overline{\mathcal{M}}_{0,\mathcal{A}}$, we let $\mathbf{q}(D)$ be the reduced image. We also conflate the branch morphism $\mathbf{br}$ and its composition with $\mathcal{R}_{\mathcal{W}} \to \overline{\mathcal{M}}_{0,\mathcal{A}}/\mathfrak{S}$.

5.1. **Boundary divisors on $\mathcal{H}_{n,\alpha}$ and $\mathcal{H}_{n,\alpha,\beta}$.** We let δ_{irr} to be the divisor (on $\mathcal{H}_{n,\alpha}$ or $\mathcal{H}_{n,\alpha,\beta}$) parameterizing covers $\mathcal{X} \xrightarrow{\varphi} \mathcal{Y}$ such that $\mathcal{X}$ has a singularity lying over a smooth point of $\mathcal{Y}$. Equivalently, a cover is in δ_{irr} if the branch divisor has a point of multiplicity at least 2. It follows that

$$\delta_{\mathrm{irr}} = \mathbf{br}^* \, \mathbf{q}(\Delta_s).$$

We let δ_{red} to be the divisor parameterizing reducible covers $\mathcal{X} \xrightarrow{\varphi} \mathcal{Y}$. A cover is reducible if and only if $\mathcal{Y}$ has an (orbi)node. In terms of divisors on $\overline{\mathcal{M}}_{0,\mathcal{A}}/\mathfrak{S}$ we have that

$$\delta_{\mathrm{red}} = \mathbf{br}^* \, \mathbf{q}(\Delta_{\mathrm{even}}) + \frac{1}{2} \mathbf{br}^* \, \mathbf{q}(\Delta_{\mathrm{odd}}).$$

On $\mathcal{H}_{n,\alpha,\beta}$ we define the *Weierstrass divisor* δ_W to be the locus of $(\varphi\colon \mathcal{X} \to \mathcal{Y}; \tau, \chi, D)$ such that χ is a ramification point of φ. Equivalently,

$$\delta_W = \mathrm{br}^* \, \mathbf{q}(\Delta_{\sigma \cap \chi}).$$

5.1.1. *Log canonical divisors on $\mathcal{H}_{n,\alpha}$.* Let $\mathcal{W} = (1, \alpha^{n+1})$. Recall that $\mathcal{R}_{\mathcal{W}}$ is the moduli space of $\mathcal{W}$-stable rational curves with a section at infinity of weight 1 and a divisor of degree $n+1$ and weight α (see Section 3). Consider the commutative diagram

$$\begin{array}{ccccc} & & \mathcal{H}_{n,\alpha} & \xrightarrow{\text{coarse}} & H_{n,\alpha} \\ & & \downarrow \mathbf{br} & \searrow^{\mathbf{br}} & \downarrow \simeq \\ \overline{\mathcal{M}}_{0,\mathcal{A}} & \xrightarrow{\mathbf{q}} & \mathcal{R}_{\mathcal{W}} & \xrightarrow{\text{coarse}} & \overline{\mathcal{M}}_{0,\mathcal{A}}/\mathfrak{S}_{n+1} = R_{\mathcal{W}}. \end{array} \tag{5.1}$$

Lemma 5.1.

(a) *The quotient morphism $\mathbf{q}\colon \overline{\mathcal{M}}_{0,\mathcal{A}} \to \overline{\mathcal{M}}_{0,\mathcal{A}}/\mathfrak{S}_{n+1}$ is simply ramified over Δ_s.*

(b) *The finite degree 2 morphism $\mathbf{br}\colon \mathcal{H}_{n,\alpha} \to \overline{\mathcal{M}}_{0,\mathcal{A}}/\mathfrak{S}_{n+1}$ is simply ramified over $\mathbf{q}(\Delta_{\mathrm{odd}})$.*

Proof. We note that (a) is standard: a quotient morphism is always ramified over the locus where the stabilizer is nongeneric (in our case, non-trivial). A local computation shows that the ramification is simple.

To prove (b), observe that the morphism **br** is ramified along the codimension one locus of hyperelliptic covers having an extra automorphism. These are precisely reducible covers possessing an odd node. By our convention, such covers map to the divisor $\mathbf{q}(\Delta_{\mathrm{odd}})$. □

Proposition 5.2. *The canonical divisor of $\mathcal{H}_{n,\alpha}$ is $K_{\mathcal{H}_{n,\alpha}} = \mathbf{br}^*(D)$, where D is the divisor on $R_{\mathcal{W}} \simeq \overline{\mathcal{M}}_{0,\mathcal{A}}/\mathfrak{S}_{n+1}$ satisfying*

$$\mathbf{q}^*(D) = \psi - \Delta_s - 2\Delta_{\mathrm{even}} - \frac{3}{2}\Delta_{\mathrm{odd}}.$$

Proof. By Lemma 5.1(b), we have $K_{\mathcal{H}_{n,\alpha}} = \mathbf{br}^* \left(K_{R_{\mathcal{W}}} + \frac{1}{2}\mathbf{q}(\Delta_{\mathrm{odd}})\right)$. By Lemma 5.1(a), we have $\mathbf{q}^* \left(K_{R_{\mathcal{W}}} + \frac{1}{2}\mathbf{q}(\Delta_{\mathrm{odd}})\right) = K_{\overline{\mathcal{M}}_{0,\mathcal{A}}} - \Delta_s + \frac{1}{2}\Delta_{\mathrm{odd}}$. It remains to recall that by Grothendieck-Riemann-Roch formula (see also [Has03, Section 3.1.1]):

$$K_{\overline{\mathcal{M}}_{0,\mathcal{A}}} = \psi_\sigma + \psi_\tau - 2\Delta = \psi - 2\Delta_{\mathrm{even}} - 2\Delta_{\mathrm{odd}}.$$

□

5.1.2. *Log canonical divisors on $\mathcal{H}_{n,\alpha,\beta}$.* The canonical divisor of $\mathcal{H}_{n,\alpha,\beta}$ is computed analogously. Let $\mathcal{W} = (1, \beta, \alpha^n)$ and $\mathcal{A} = (1, \beta, \underbrace{\alpha, \dots, \alpha}_{n})$, and consider the diagram

$$\begin{array}{ccccc} & & \mathcal{H}_{n,\alpha,\beta} & \xrightarrow{\text{coarse}} & H_{n,\alpha,\beta} \\ & & \downarrow \mathbf{br} & \searrow^{\mathbf{br}} & \downarrow \simeq \\ \overline{\mathcal{M}}_{0,\mathcal{A}} & \xrightarrow{\mathbf{q}} & \mathcal{R}_{\mathcal{W}} & \xrightarrow{\text{coarse}} & \overline{\mathcal{M}}_{0,\mathcal{A}}/\mathfrak{S}_n = R_{\mathcal{W}}. \end{array} \tag{5.2}$$

The only difference is that the morphism $\mathbf{br}\colon \mathcal{H}_{n,\alpha,\beta} \to R_{\mathcal{W}}$ is now ramified over $\mathbf{q}(\Delta_{\text{odd}})$ *and* $\mathbf{q}(\Delta_{\sigma\cap\chi})$. Therefore, $K_{\mathcal{H}_{n,\alpha,\beta}} = \mathbf{br}^*\left(K_{R_{\mathcal{W}}} + \frac{1}{2}\mathbf{q}(\Delta_{\text{odd}} + \Delta_{\sigma\cap\chi})\right)$. The quotient morphism $\mathbf{q}\colon \overline{\mathcal{M}}_{0,\mathcal{A}} \to \overline{\mathcal{M}}_{0,\mathcal{A}}/\mathfrak{S}_n$ is still ramified only at Δ_s. It follows that

$$\begin{aligned} \mathbf{q}^*\left(K_{R_{\mathcal{W}}} + \frac{1}{2}\mathbf{q}(\Delta_{\text{odd}} + \Delta_{\sigma\cap\chi})\right) &= K_{\overline{\mathcal{M}}_{0,\mathcal{A}}} - \Delta_s + \frac{1}{2}(\Delta_{\text{odd}} + \Delta_{\sigma\cap\chi}) \\ &= \psi_\tau + \psi_\sigma + \psi_\chi - \Delta_s - 2\Delta_{\text{even}} - \frac{3}{2}\Delta_{\text{odd}} + \frac{1}{2}\Delta_{\sigma\cap\chi}. \end{aligned}$$

We summarize these computations in the following proposition.

Proposition 5.3. *The canonical divisor of $\mathcal{H}_{n,\alpha,\beta}$ is $K_{\mathcal{H}_{n,\alpha,\beta}} = \mathbf{br}^*(D)$, where D is the divisor on $R_{\mathcal{W}} \simeq \overline{\mathcal{M}}_{0,\mathcal{A}}/\mathfrak{S}_n$ satisfying*

$$\mathbf{q}^*(D) = \psi - \Delta_s - 2\Delta_{\text{even}} - \frac{3}{2}\Delta_{\text{odd}} + \frac{1}{2}\Delta_{\sigma\cap\chi}.$$

5.2. **Log canonical models of $\mathcal{T}_{A_n}$.** Fix an integer $n \geq 2$. In Section 6, we show that for $k' \geq k$ there exists a natural reduction morphism $\mathcal{H}_n[k] \to \mathcal{H}_n[k']$. If k' is even (resp. odd), the morphism $\mathcal{H}_n[k] \to \mathcal{H}_n[k']$ replaces a tail of genus $k'/2$ (resp. a bridge of genus $(k'-1)/2$) by an $A_{k'}$ singularity. We obtain a sequence of birational contractions

$$\mathcal{H}_n[1] \to \mathcal{H}_n[2] \to \dots \to \mathcal{H}_n[n-1],$$

where the initial space $\mathcal{H}_n[1]$ is the stack of Harris-Mumford *admissible covers* of genus $\left\lfloor \frac{n}{2} \right\rfloor$ (recall that admissible covers have at worst A_1 singularities), and, by Example 4.7, the final space is

$$\mathcal{H}_n[n-1] \simeq [\mathrm{Def}(A_n) \smallsetminus \mathbf{0} \;/\; \mathbb{G}_m] \simeq \begin{cases} \mathcal{P}(2, 3, \dots, n+1), & \text{if } n \text{ is odd,} \\ \mathcal{P}(4, 6, \dots, 2n+2), & \text{if } n \text{ is even.} \end{cases}$$

(Note that the generic stabilizer of $\mathcal{H}_n[n-1]$ is μ_2 in the case of even n.)

In what follows, we describe the intermediate spaces as log canonical models of $\mathcal{H}_n[1]$.

Proposition 5.4. *The divisor $K_{\mathcal{H}_{n,\alpha}} + (\alpha+1/2)\delta_{irr} + \delta_{red}$ is a pullback of an ample divisor from the coarse moduli space.*

Proof. By Proposition 5.2, we have $K_{\mathcal{H}_{n,\alpha}} + (\alpha + 1/2)\delta_{\text{irr}} + \delta_{\text{red}} = \mathbf{br}^*(D)$, where D is the divisor on $\overline{\mathcal{M}}_{0,\mathcal{A}}/\mathfrak{S}_{n+1}$ satisfying

$$\begin{aligned} \mathbf{q}^*(D) &= (\psi_\tau + \psi_\sigma - \Delta_s - 2\Delta_{\text{even}} - \frac{3}{2}\Delta_{\text{odd}}) + (2\alpha+1)\Delta_s + (\Delta_{\text{even}} + \frac{1}{2}\Delta_{\text{odd}}) \\ &= \psi_\tau + \psi_\sigma + 2\alpha\Delta_s - \Delta_n. \end{aligned}$$

By [Fed11, Theorem 5.1], the divisor $\psi_\tau + \psi_\sigma + 2\alpha\Delta_s - \Delta_n$ is ample on $\overline{\mathcal{M}}_{0,\mathcal{A}}$. □

We obtain the following result which finishes the proof of Part 5 of Main Theorem 1.

Theorem 5.5. *Let $k \in \{1, \dots, n-1\}$, then for any $\alpha \in \left(\frac{1}{2} + \frac{1}{k+2}, \frac{1}{2} + \frac{1}{k+1}\right] \cap \mathbb{Q}$*

$$H_n[k] = \operatorname{Proj} R\left(\mathcal{H}_n[1], K_{\mathcal{H}_n[1]} + \alpha\delta_{irr} + \delta_{red}\right).$$

In other words, the coarse moduli space of $\mathcal{H}_n[k]$ is a log canonical model of $\mathcal{H}_n[1]$.

Proof. By Proposition 5.4, the divisor $K_{\mathcal{H}_{n,\alpha}} + \alpha\delta_{\mathrm{irr}} + \delta_{\mathrm{red}}$ is a pullback of an ample divisor from the coarse moduli space. Let $f\colon \mathcal{H}_n[1] \to \mathcal{H}_n[k]$ be the reduction morphism, then the discrepancy

$$K_{\mathcal{H}_n[1]} + \alpha\delta_{\mathrm{irr}} + \delta_{\mathrm{red}} - f^*(K_{\mathcal{H}_n[k]} + \alpha\delta_{\mathrm{irr}} + \delta_{\mathrm{red}})$$

is easily seen to be effective for $\alpha \le \frac{k+3}{2(k+1)}$ (cf. Lemma 6.4). The statement now follows by [HH09, Proposition A.13]. □

5.3. **Log canonical models of $\mathcal{T}_{D_n}$.** Fix a number $n \ge 4$. In Section 6, we construct a natural reduction morphism $\mathcal{H}_n[k, \ell] \to \mathcal{H}_n[k', \ell']$ (for $k' \ge k$ and $\ell' \ge \ell$) such that

- If k' is even (resp. odd), then any tail of genus $k'/2$ (resp. a bridge of genus $(k'-1)/2$) is replaced by an $A_{k'}$ singularity.
- If ℓ' is even (resp. odd), then any pointed bridge of genus $\ell'/2 - 1$ (resp. a pointed tail of genus $(\ell'-1)/2$) is replaced by a $D_{\ell'}$ singularity.

We obtain a lattice of moduli stacks, where each arrow is a divisorial contraction:

$$\begin{array}{ccccccccc}
\mathcal{H}_n[1,1] & \longrightarrow & \mathcal{H}_n[1,2] & & & & & & \\
\downarrow & & \downarrow & & & & & & \\
\mathcal{H}_n[2,1] & \longrightarrow & \mathcal{H}_n[2,2] & \longrightarrow & \mathcal{H}_n[2,3] & & & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
\cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{H}_n[n-2,1] & \longrightarrow & \mathcal{H}_n[n-2,2] & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \mathcal{H}_n[n-2,n-1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{H}_n[n-1,1] & \longrightarrow & \mathcal{H}_n[n-1,2] & \longrightarrow & \cdots & \longrightarrow & \mathcal{H}_n[n-1,n-2] & \longrightarrow & \mathcal{H}_n[n-1,n-1]
\end{array}$$

$\mathcal{H}_n[1,1]$ is the stack of 1-pointed admissible covers of genus $\lfloor \frac{n-1}{2} \rfloor$ and by Example 4.7

$$\mathcal{H}_n[n-1,n-1] \simeq [\operatorname{Def}(D_n) \smallsetminus \mathbf{0}/\mathbb{G}_m] \simeq \begin{cases} \mathcal{P}(n/2, 1, 2, 3, \dots, n-1), & \text{if } n \text{ is even,} \\ \mathcal{P}(n, 2, 4, 6, \dots, 2n-2), & \text{if } n \text{ is odd.} \end{cases}$$

We now describe the intermediate spaces as log canonical models of $\mathcal{H}_n[1,1]$. For this we look at special log canonical divisors described in the following proposition.

Proposition 5.6. *The divisor $K_{\mathcal{H}_{n,\alpha,\beta}} + (\alpha + 1/2)\delta_{irr} + \delta_{red} + (2\alpha + 2\beta - 1)\delta_W$ is a pullback of an ample divisor from the coarse moduli space.*

Proof. By Proposition 5.3, we have $K_{\mathcal{H}_{n,\alpha,\beta}}+(\alpha+1/2)\delta_{\rm irr}+\delta_{\rm red}+(2\alpha+2\beta-1)\delta_W=\mathbf{br}^*(D)$ where D is the divisor on $\overline{\mathcal{M}}_{0,\mathcal{A}}/\mathfrak{S}_n$ satisfying

$$\begin{aligned}\mathbf{q}^*(D)&=(\psi-\Delta_s-2\Delta_{\rm even}-\frac{3}{2}\Delta_{\rm odd}+\frac{1}{2}\Delta_{\sigma\cap\chi})+(2\alpha+1)\Delta_s\\&\quad+(\Delta_{\rm even}+\frac{1}{2}\Delta_{\rm odd})+\frac{1}{2}(2\alpha+2\beta-1)\Delta_{\sigma\cap\chi}\\&=\psi_\tau+\psi_\sigma+\psi_\chi-\Delta_n+2\alpha\Delta_s+(\alpha+\beta)\Delta_{\sigma\cap\chi}.\end{aligned}$$

By [Fed11, Theorem 5.1], applied to the weight vector $\mathcal{A}=(1,\beta,\underbrace{\alpha,\dots,\alpha}_{n})$, the divisor

$$\psi_\tau+\psi_\sigma+\psi_\chi-\Delta_n+2\alpha\Delta_s+(\alpha+\beta)\Delta_{\sigma\cap\chi}$$

is ample on $\overline{\mathcal{M}}_{0,\mathcal{A}}$. □

Proposition 5.6 together with Lemma 6.4 and [HH09, Proposition A.13] finish the proof of Part 5 of Main Theorem 2. Specializing to the case of $\alpha+\beta=1/2$, we obtain

Theorem 5.7. *For $k\in\{2,\dots,2n-4\}$, set $\ell=\lfloor k/2\rfloor+1$. Note that $\mathcal{H}_{n,1/k,(k-2)/2k}=\mathcal{H}_n[k-1,\ell]$ if $k\le n$, and $\mathcal{H}_{n,1/k,(k-2)/2k}=\mathcal{H}_n[n-1,\ell]$ if $k>n$. The coarse moduli space of $\mathcal{H}_{n,1/k,(k-2)/2k}$ is*

$$\operatorname{Proj}R\big(\mathcal{H}_n[1,1],K_{\mathcal{H}_n[1,1]}+(1/2+1/k)\delta_{irr}+\delta_{red}\big).$$

Remark 5.8. Theorem 5.7 says that curves with A_{k-1} and D_ℓ singularities first appear in the moduli space $\operatorname{Proj}R\left(\mathcal{H}_n[1,1],K_{\mathcal{H}_n[1,1]}+\alpha\delta_{\rm irr}+\delta_{\rm red}\right)$ for α that are exactly the log canonical thresholds of the discriminant inside the deformation spaces $\operatorname{Def}(A_{k-1})$ and $\operatorname{Def}(D_\ell)$ (see Section 2.1.2).

6. Reduction morphisms

In this section, we establish the existence of natural reduction morphisms between different $\mathcal{H}_n[k,\ell]$. Recall from Section 4 that $\mathcal{H}_n[k,\ell]=\mathcal{H}_{n,\alpha,\beta}$ parameterizes pointed quasi-admissible covers $(\varphi\colon\mathcal{X}\to\mathcal{Y};\tau,\chi,D)$, where the branch divisor D has degree n and weight α, and the section χ has weight β. The integers k and ℓ are determined uniquely by the inequalities

$$\begin{aligned}\frac{1}{k+2}&<\alpha\le\frac{1}{k+1},\\1-(\ell+1)\alpha&<\beta\le1-\ell\alpha.\end{aligned}\tag{6.1}$$

The starting point for us will be the existence of reduction morphisms between moduli stacks $\mathcal{R}_{\mathcal{W}}$ of divisorially marked rational curves (see Definition 3.2). Namely, if $\mathcal{W}=(1,\beta,\alpha^n)$ and $\mathcal{W}'=(1,\beta',(\alpha')^n)$ are weight vectors such that $\alpha\ge\alpha'$ and $\beta\ge\beta'$, then by [Has03, Theorem 4.1][1] there exists the reduction morphism $r_{\mathcal{W},\mathcal{W}'}\colon\mathcal{R}_{\mathcal{W}}\to\mathcal{R}_{\mathcal{W}'}$. Before we proceed we make a useful observation.

Lemma 6.1. *Let (k,ℓ) and (k',ℓ') be the integers associated to weight vectors $\mathcal{W}$ and $\mathcal{W}'$ and uniquely determined by Inequalities (6.1). If $k\le k'$ and $\ell\le\ell'$, then the reduction morphism $r_{\mathcal{W},\mathcal{W}'}\colon\mathcal{R}_{\mathcal{W}}\to\mathcal{R}_{\mathcal{W}'}$ exists. When this happens we say that $(k,\ell)\preceq(k',\ell')$.*

[1]The proof in [Has03] is given for weighted pointed curves but generalizes word for word to the situation of divisorially marked curves.

Proof. The lemma says that the morphism $r_{\mathcal{W},\mathcal{W}'}$ exists even when $\beta' > \beta$ or $\alpha' > \alpha$ so long as the condition $(k,\ell) \preceq (k',\ell')$ is satisfied. The idea of proof is clear: The $\mathcal{W}$-stability is a condition on how many points of weight α and β can come together. Thus $\mathcal{R}_{\mathcal{W}}$ does not change as long as k and ℓ remain constant. We construct the reduction morphism by tweaking the weights so that $\mathcal{R}_{\mathcal{W}}$ and $\mathcal{R}_{\mathcal{W}'}$ are unchanged but [Has03, Theorem 4.1] now applies.

It suffices to consider the cases $(k',\ell') = (k+1,\ell)$ and $(k',\ell') = (k,\ell+1)$. In the former case, we take $\alpha = 1/(k+2)+\varepsilon$, $\alpha' = 1/(k+2)$, $\beta = 1-\ell\alpha$, $\beta' = 1-(\ell+1)\alpha'+\delta$, where $0 < \varepsilon, \delta \ll 1$. Then $\alpha > \alpha'$ and $\beta > \beta'$, so the reduction morphism exists. In the latter case, we take $\alpha = \alpha' = 1/(k+1)$, $\beta = 1-\ell\alpha$, $\beta' = 1-(\ell+1)\alpha'$. Then $\alpha = \alpha'$ and $\beta > \beta'$, so the reduction morphism exists. ☐

Theorem 6.2. *Let $\mathcal{W} = (1,\beta,\alpha^n)$ and $\mathcal{W}' = (1,\beta',(\alpha')^n)$ be weight vectors with $(k,\ell) \preceq (k',\ell')$. Then there exists a natural birational* reduction morphism $r_{\mathcal{W},\mathcal{W}'}\colon \mathcal{H}_n[k,\ell] \to \mathcal{H}_n[k',\ell']$.

The case of $\mathcal{H}_n[k]$ is simpler. For completeness, we include the statement here without a proof.

Theorem 6.3. *Let $\mathcal{W} = \big(1,\alpha^{n+1}\big)$ and $\mathcal{W}' = \big(1,(\alpha')^{n+1}\big)$, where $\alpha \in (1/(k+2), 1/(k+1)]$ and $\alpha' \in (1/(k'+2), 1/(k'+1)]$ for $k' > k$. Then there exists a natural birational* reduction morphism

$$r_{\mathcal{W},\mathcal{W}'}\colon \mathcal{H}_n[k] \to \mathcal{H}_n[k'].$$

Proof of Theorem 6.2: As in Lemma 6.1, we can assume that either $(k',\ell') = (k+1,\ell)$ or $(k',\ell') = (k,\ell+1)$. Moreover, we can choose weights $\alpha \geq \alpha'$ and $\beta \geq \beta'$ without changing $\mathcal{H}_n[k,\ell]$ and $\mathcal{H}_n[k',\ell']$.

Denote $\mathcal{H} := \mathcal{H}_n[k,\ell]$ and $\mathcal{H}' := \mathcal{H}_n[k',\ell']$. Since $\mathcal{H}$ is a smooth proper Deligne-Mumford stack over $\mathbb{K}$ by Theorems 4.9 and 4.10, there exists a surjective étale morphism $T \to \mathcal{H}$ with T a smooth scheme. We will construct a morphism $f\colon T \to \mathcal{H}'$ such that for $T \times_{\mathcal{H}} T \overset{\mathrm{pr}_1,\mathrm{pr}_2}{\longrightarrow} T$ the two morphisms $f \circ \mathrm{pr}_1$ and $f \circ \mathrm{pr}_2$ are isomorphic and the isomorphism satisfies the cocycle condition on $T \times_{\mathcal{H}} T \times_{\mathcal{H}} T$. By descent, this will define a morphism $\mathcal{H} \to \mathcal{H}'$.

To begin, let $(\varphi\colon \mathcal{X} \to \mathcal{Y}; \tau, \chi, D)$ be the exhausting family over T. By Lemma 4.3, $\mathcal{X} \simeq \mathit{Spec}_{\mathcal{Y}}\big(\mathcal{O}_{\mathcal{Y}} \oplus \mathcal{L}^{-1}\big)$, where $\mathcal{L}^2 = \mathcal{O}_{\mathcal{Y}}(D)$. Let $\rho\colon \mathcal{Y} \to Y$ be the morphism to the coarse moduli space. By definition, Y is a family of $\mathcal{W}$-stable divisorially marked rational curves. By [Has03, Theorem 4.1], we can form a T-morphism $\xi\colon Y \to Y'$ such that Y' is a $\mathcal{W}'$-stable rational curve. Importantly, the formation of ξ commutes with base change. The reduction morphism ξ contracts all rational tails in the fibers of $Y \to T$ on which $\omega_{Y/T}(\tau + \alpha' D + \beta'\chi)$ has non-positive degree to smooth points in the fibers of $Y' \to T$. Denote by E the union of the contracted curves. Then ξ is an isomorphism away from E.

Set $U := Y' - \mathrm{Sing}(Y'/T)$ and $V := Y' - \xi(E)$. Then U and V form an open cover of Y'. Since $\rho\colon \rho^{-1}(V) \to V$ is an isomorphism, we can construct a T-orbicurve $\mathcal{Y}'$ by gluing U and $\rho^{-1}(V)$ along $U \cap V$. In words, $\mathcal{Y}'$ has the same stack structure as $\mathcal{Y}$ away from $\xi(E)$ and is isomorphic to Y' in a Zariski neighborhood of $\xi(E)$. The morphism $\mathcal{Y}' \to \mathcal{Y}$ will also be denoted by ξ.

Since T is smooth, we have that $\mathcal{Y}'$ is a smooth scheme in a neighborhood of $\xi(E)$. We also have $\mathrm{codim}(\xi(E), \mathcal{Y}') \geq 2$ and $\mathrm{codim}(\mathrm{Sing}(\mathcal{Y}'/T), \mathcal{Y}') \geq 2$. Therefore, the line bundle

$\mathcal{L}$ and the Cartier divisor D uniquely extend from $U \cap V$ to $\mathcal{Y}'$. Denote these extensions by $\mathcal{L}'$ and D'. Clearly, $(\mathcal{L}')^2 = \mathcal{O}_{\mathcal{Y}'}(D')$. Lastly, define sections $\tau' = \xi \circ \tau \colon T \to \mathcal{Y}'$ and $\chi' = \xi \circ \chi \colon T \to \mathcal{Y}'$.

By construction, $\mathcal{Y}'$ (marked by τ', χ', and D') is a $\mathcal{W}'$-stable even rational orbicurve over T. Since the formation of Y' commutes with base change, so does the formation of $\mathcal{Y}'$. Same holds for the formation of τ' and χ'. Finally, the formation of D' commutes with *smooth* base change. It follows that $(\mathcal{Y}'; \tau', \chi', D')$ is a $\mathcal{W}'$-stable even rational orbicurve and there is an isomorphism $\mathrm{pr}_1^* \mathcal{Y}' \simeq \mathrm{pr}_2^* \mathcal{Y}'$ of even rational orbicurves over $T \times_{\mathcal{H}} T$ whose pullbacks satisfy the cocycle condition on $T \times_{\mathcal{H}} T \times_{\mathcal{H}} T$.

We now define $\mathcal{X}' := \mathit{Spec}_{\mathcal{Y}'}(O_{\mathcal{Y}'} \oplus (\mathcal{L}')^{-1})$. From the construction, $\xi_*(\mathcal{O}_{\mathcal{Y}} \oplus \mathcal{L}^{-1}) = O_{\mathcal{Y}'} \oplus (\mathcal{L}')^{-1}$. It follows that $\xi_*(\varphi_* \mathcal{O}_{\mathcal{X}}) = O_{\mathcal{Y}'} \oplus (\mathcal{L}')^{-1}$. The induced morphism $\mathcal{X} \to \mathcal{X}'$ is such that $\mathcal{X} \to \mathcal{X}' \to \mathcal{Y}'$ is the Stein factorization of $\mathcal{X} \to \mathcal{Y}'$. Clearly, $\mathcal{X} \to \mathcal{X}'$ contracts those components in the fibers of $\mathcal{X} \to T$ which are not $\mathcal{W}'$-stable. Finally, if τ is even we define sections $\tau_1, \tau_2 \colon T \to \mathcal{X}'$ as compositions of $\tau_1, \tau_2 \colon T \to \mathcal{X}$ and $\mathcal{X} \to \mathcal{X}'$. This finishes the construction of a pointed quasi-admissible cover over T inducing a morphism $f \colon T \to \mathcal{H}'$. Since all of the steps in the construction are canonical and commute with smooth base change we conclude that f descends to a morphism $\mathcal{H} \to \mathcal{H}'$, as required. $\square$

6.1. **Local structure of reduction morphisms.** We describe the exceptional loci $\mathrm{Exc}(f)$ and $\mathrm{Exc}(g)$ of $f \colon \mathcal{H}_n[k, \ell] \to \mathcal{H}_n[k+1, \ell]$ and $g \colon \mathcal{H}_n[k, \ell] \to \mathcal{H}_n[k, \ell+1]$, the remaining cases being analogous.

From the construction, $\mathrm{Exc}(f)$ is the locus of reducible quasi-admissible covers $(\varphi \colon \mathcal{X} \to \mathcal{Y}; \tau, \chi, D)$ such that the target Y has a rational component R meeting the rest of Y in a single point, disjoint from τ and $\varphi(\chi)$, and such that $\deg D|_R = k+2$. If k is odd, the component of X lying over R is a hyperelliptic curve of genus $(k+1)/2$ meeting the rest of X in two conjugate points; we say that the cover has a *hyperelliptic tail of genus* $(k+1)/2$. If k is even, the component of X lying over R is a hyperelliptic curve of genus $k/2$ meeting the rest of X in a single Weierstrass point; we say that the cover has a *hyperelliptic bridge of genus* $k/2$. In an étale neighborhood of $(\varphi \colon \mathcal{X} \to \mathcal{Y}; \tau, \chi, D) \in \mathrm{Exc}(f)$, the exceptional divisor is the locus of deformations preserving a node of Y. Thus $\mathrm{Exc}(f)$ is a Cartier divisor on $\mathcal{H}_n[k, \ell]$. By the above, there is also an isomorphism $\mathrm{Exc}(f) \simeq \mathcal{H}_{k+1}[k]$.

Similarly, $\mathrm{Exc}(g)$ is the divisor of reducible quasi-admissible covers $(\varphi \colon \mathcal{X} \to \mathcal{Y}; \tau, \chi, D)$ such that Y has a rational component R meeting the rest of Y in a single point, disjoint from τ, and such that $\deg D|_R = \ell+1$ and $\varphi(\chi) \in R$. If ℓ is odd, the component of X lying over R is a hyperelliptic curve of genus $(\ell-1)/2$ meeting the rest of X in two conjugate points and marked by χ; we say that the cover has a *pointed hyperelliptic bridge of genus* $(\ell-1)/2$. If ℓ is even, the component of X lying over R is a hyperelliptic curve of genus $\ell/2$ meeting the rest of X in a single Weierstrass point and marked by χ; we say that the cover has a *pointed hyperelliptic tail of genus* $\ell/2$. Evidently, there is an isomorphism $\mathrm{Exc}(g) \simeq \mathcal{H}_{\ell+1}[k, \ell]$.

Using the functorial interpretation of morphisms f and g given above, a routine computation with explicit test families gives the following result.

Lemma 6.4. *In the $\mathbb{Q}$-Picard group of $\mathcal{H}_n[k,\ell]$, we have*

$$K_{\mathcal{H}_n[k,\ell]} + (\alpha+1/2)\delta_{irr} + \delta_{red} + (2\alpha+2\beta-1)\delta_W$$
$$- f^*(K_{\mathcal{H}_n[k+1,\ell]} + (\alpha+1/2)\delta_{irr} + \delta_{red} + (2\alpha+2\beta-1)\delta_W) = (1-(k+2)\alpha)\operatorname{Exc}(f)$$

and

$$K_{\mathcal{H}_n[k,\ell]} + (\alpha+1/2)\delta_{irr} + \delta_{red} + (2\alpha+2\beta-1)\delta_W$$
$$- g^*(K_{\mathcal{H}_n[k,\ell+1]} + (\alpha+1/2)\delta_{irr} + \delta_{red} + (2\alpha+2\beta-1)\delta_W) = (1-(\ell+1)\alpha-\beta)\operatorname{Exc}(g).$$

7. Deformations of D singularities

In this section, we explain how to view pointed quasi-admissible covers of Definition 4.2 as curves with D singularities. More precisely, we establish equivalence between deformations of an A_{n-1} singularity with a section and a D_n singularity.

To begin, let $X = \operatorname{Spec} \mathbb{K}[[x,y]]/(y^2-x^n)$ be the A_{n-1} singularity and $s=(0,0)\in X$ be a section. Consider the local deformation functor $\operatorname{Def}(X,s)$ which sends an Artinian $\mathbb{K}$-algebra A with the residue field $\mathbb{K}$ to the set of isomorphism classes of Cartesian diagrams

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow\uparrow s & & \downarrow\uparrow \Sigma \\ \operatorname{Spec}\mathbb{K} & \longrightarrow & \operatorname{Spec} A \end{array}$$

Theorem 7.1. *The functor* $\operatorname{Def}(X,s)$ *is naturally isomorphic to* $\operatorname{Def}(D_n)$*, the local deformation functor of the* D_n *singularity* $\operatorname{Spec}\mathbb{K}[[x,u]]/(x(u^2-x^{n-2}))$.

Proof. To prove the statement, it suffices to establish an isomorphism between miniversal deformation spaces of $\operatorname{Def}(X,s)$ and $\operatorname{Def}(D_n)$. To begin, a miniversal deformation space of an isolated planar singularity $\mathbb{K}[[x,y]]/(f(x,y))$ with a section $(x,y)=(0,0)$ can be taken to be the $\mathbb{K}$-vector space $\mathfrak{m}/(f, \mathfrak{m}\cdot(\partial f/\partial x, \partial f/\partial y))$, where $\mathfrak{m}=(x,y)\subset \mathbb{K}[[x,y]]/(f(x,y))$ is the maximal ideal (see [KM04, Section 2] or [MvS01, Lemma 2.1]). Moreover, if monomials x^iy^j form a basis of the said $\mathbb{K}$-vector space, then

$$\operatorname{Spec}\mathbb{K}[[x,y,\{t_{ij}\}]]/(f(x,y)-\sum_{i,j} t_{ij}x^iy^j) \to \operatorname{Spec}\mathbb{K}[[\{t_{ij}\}]]$$

is the miniversal deformation with the universal section $\Sigma : \{x=y=0\}$. Applying this to $f(x,y)=y^2-x^n$, we obtain that $T := \mathbb{K}[b, a_0, \dots, a_{n-2}]$ is the base of the miniversal deformation of $\operatorname{Def}(X,s)$ and

$$\mathcal{X} := \{y^2 - by - (x^n + a_{n-2}x^{n-1} + \cdots + a_0x) = 0\} \subset \mathbb{A}^2_{x,y}\times T$$

is the miniversal family.

Next, let $\Sigma : \{x=0, y=0\}$ be the universal section, and $\Sigma' : \{x=0, y-b=0\}$ be the conjugate section. Since Σ' is not a Cartier divisor on the total family $\mathcal{X}$, we can blow-up Σ' to obtain a new family $\mathcal{Y} := \operatorname{Bl}_{\Sigma'}\mathcal{X}$ of plane curves over T. By construction, we can regard $\mathcal{Y}$ as a subvariety of $\mathbb{P}^1_{[U:V]}\times\mathbb{A}^2_{x,y}\times T$. We proceed to describe $\mathcal{Y}$ by explicit equations: Where $V\neq 0$, we have $y-b=xU/V$ and the equation of $\mathcal{Y}$ in terms of x and $u:=U/V$ is

$$xu^2 + ub - (x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0) = 0. \tag{7.1}$$

By the discussion in Section 2.1.1, Equation (7.1) defines precisely the miniversal deformation of the D_n singularity $x(u^2 - x^{n-2}) = 0$. In particular, the central fiber of $\mathcal{Y} \to T$ has a unique singularity of type D_n, and the family $\mathcal{Y} \to T$ is its miniversal deformation. This finishes the proof.

We note that, more generally, the blow-up $\mathcal{Y} \to \mathcal{X}$ replaces a fiber of $\mathcal{X} \to T$ in which the section Σ coincides with an A_{k-1} singularity by a fiber of $\mathcal{Y} \to T$ with a D_k singularity. This effect on fibers is illustrated in Figure 2. □

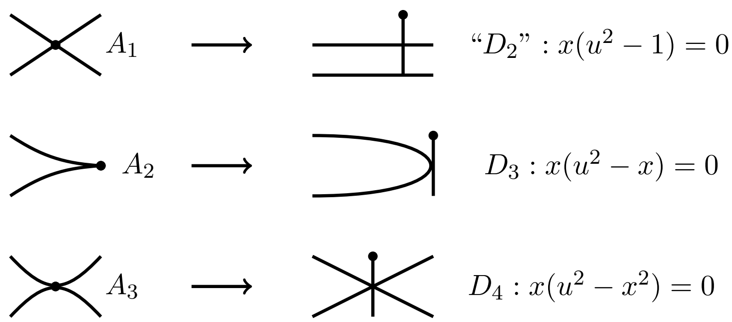


FIGURE 2. Replacing an A_{k-1} singularity with a section by a D_k singularity.

8. Applications

8.1. **(A, D)-stable reduction.** Given a proper connected curve C of arithmetic genus $g \gg 0$ with a single isolated smoothable singularity p, one has a rational *moduli map* $j\colon \operatorname{Def}(C) \dashrightarrow \overline{\mathcal{M}}_g$. The problem of resolving the indeterminacy of j plays an important role in the study of alternate compactifications of $\mathcal{M}_g$ (see Section 1).

An application of our Main Theorem 2 is an iterative functorial resolution of the indeterminacy of the moduli map $j\colon \operatorname{Def}(C) \dashrightarrow \overline{\mathcal{M}}_g$ in the case when C has only A and D singularities.

Theorem 8.1 ((A_k, D_ℓ)-stable reduction)**.** *Suppose $\mathcal{C} \to \operatorname{Def}(D_n)$ is the miniversal deformation ($n \geq 4$). Then for integers k and ℓ such that $\ell \leq \min\{k+1, n\}$, there is a representable by proper Deligne-Mumford stacks morphism $f_{k,\ell}\colon T \to \operatorname{Def}(D_n)$, which is an isomorphism away from $\Delta \subset \operatorname{Def}(D_n)$, such that $\mathcal{C}|_{\operatorname{Def}(D_n) \smallsetminus \Delta}$ extends to the family of curves with at worst A_k and D_ℓ singularities over T.*

Proof. To begin, consider the moduli stack $\mathcal{H}_{n+1}[n, n]$ with the universal quasi-admissible cover $\varphi\colon \mathcal{X} \to \mathcal{Y}$. We restrict to a neighborhood of the distinguished point in $\mathcal{H}_{n+1}[n, n]$ corresponding to the unique quasi-admissible cover with a D_n singularity (cf. Theorem 7.1). By Proposition 4.13 , there is an étale neighborhood U of this point that is isomorphic to $\operatorname{Def}(D_n)$. Moreover, the restriction of $\mathcal{X} \to \mathcal{H}_{n+1}[n, n]$ to U is the miniversal deformation of D_n, and if $\delta := \delta_{\mathrm{irr}} \cup \delta_{\mathrm{red}} \cup \delta_W \subset \mathcal{H}_{n+1}[n, n]$, then $U \cap \delta = \Delta \subset \operatorname{Def}(D_n)$.

Consider now the reduction morphism $f\colon \mathcal{H}_{n+1}[k, \ell] \to \mathcal{H}_{n+1}[n, n]$. By construction, f is an isomorphism over $\mathcal{H}_{n+1}[n, n] \smallsetminus \delta$ and the universal cover over $\mathcal{H}_{n+1}[k, \ell]$ agrees with $\mathcal{X}$ over $\mathcal{H}_{n+1}[n, n] \smallsetminus \delta$. By definition, the universal cover over $\mathcal{H}_{n+1}[k, \ell]$ is a family of quasi-admissible covers with at worst A_k and D_ℓ singularities. Passing to the relative coarse moduli space over $\mathcal{H}_{n+1}[k,\ell]$ (this is possible because the characteristic is 0), we obtain a family of curves with at worst A_k and D_ℓ singularities extending the miniversal family over $\mathrm{Def}(D_n) \smallsetminus \Delta$. It follows that

$$f_{k,\ell} := f|_{f^{-1}(U)} \colon f^{-1}(U) \to U \simeq \mathrm{Def}(D_n),$$

is a requisite morphism. □

We note that in practice one desires an explicit blow-up procedure that, for an arbitrary family of curves, allows one to replace A_k singularities in the fibers by hyperelliptic tails with at worst A_{k-1} singularities, and to replace D_ℓ singularities in the fibers by hyperelliptic tails with at worst $A_{\ell-1}$ and $D_{\ell-1}$ singularities. Theorem 7.1 (and explicit blow-ups presented in its proof) reduces the problem to that for A singularities only and the following proposition (essentially generalizing [Fed07, Section 5.1] to the case of even k) is an illustration of Theorem 8.1 for the A case.

Proposition 8.2. *For a miniversal family $\mathcal{C} \to T$ of an A_k-singularity ($y^2 = x^{k+1}$) there is an alteration $f \colon T' \to T$ and a weighted blow-up $\mathcal{C}' \to \mathcal{C} \times_T T'$ of the A_k-locus in the fibers such that $\mathcal{C}' \to T'$ is a flat family of curves with at worst A_{k-1} singularities and such that $\mathcal{C}'|_{f^{-1}(0)} \simeq \mathcal{Y}_1 \cup \mathcal{Y}_2$ is a union of two irreducible components such that $\mathcal{Y}_1 \to f^{-1}(0)$ is an isotrivial family of normalizations of the central fiber $\mathcal{C}_0$ and $\mathcal{Y}_2 \to f^{-1}(0)$ is a family of curves in $\mathcal{H}_k[k-1]$.*

Proof. We can assume that $T \simeq \operatorname{Spec} \mathbb{K}[a_0, a_1, \dots, a_{k-1}]$ and the miniversal family $\mathcal{C}$ is given by the equation

$$y^2 = x^{k+1} + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0.$$

We begin with a finite base change $a_i = b_i^{k+1-i}$. Set T' to be the blow-up of $\operatorname{Spec} \mathbb{K}[b_0, \dots, b_{k-1}]$ along the ideal $(b_0, \dots, b_{k-1})$ and denote the resulting morphism $T' \to T$ by f. By construction, f is a composition of a faithfully flat finite and a proper birational morphisms. Consider the customary affine cover $T' = \bigcup_{j=0}^{k-1} U_j$ where $U_j = \operatorname{Spec} \mathbb{K}[u, c_0, \dots, \hat{c_j}, \dots, c_{k-1}]$. Then the morphism $U_j \to T$ is given by

$$b_i \mapsto uc_i \ \text{ for } \ i \neq j \text{ and } b_j \mapsto u.$$

Note that the exceptional divisor $E := f^{-1}(0)$ of f is defined by the equation $u = 0$ on U_j.

We claim that performing a weighted blow-up of $\mathcal{C} \times_T T'$ with $\mathrm{weight}(x, y, u) = (2, k+1, 2)$, or, to put it differently, taking $\mathcal{C}' := \mathrm{Bl}_{\mathcal{J}}(\mathcal{C} \times_T T')$ where $\mathcal{J} = ((\mathcal{I}_E, x)^{(k+1)/2}, y)$ if k is odd, and $\mathcal{J} = ((\mathcal{I}_E, x)^{k+1}, y^2, y(\mathcal{I}_E, x)^{k/2+1})$ if k is even, gives us a requisite family. To begin, denote the exceptional divisor of $\mathcal{C}' = \mathrm{Bl}_{\mathcal{J}}(\mathcal{C} \times_T T') \to \mathcal{C} \times_T T'$ by $\mathcal{Y}_2$ and the strict transform of $\mathcal{C}_0 \times f^{-1}(0)$ by $\mathcal{Y}_1$. To check the assertion, we work over the affine patch $U_j = \operatorname{Spec} \mathbb{K}[u, c_0, \dots, \hat{c_j}, \dots, c_{k-1}]$ over which the equation of $\mathcal{C} \times_T T'$ is

$$y^2 = x^{k+1} + c_{k-1}^2 u^2 x^{k-1} + \cdots + u^j x^{k+1-j} + \cdots + c_0^{k+1} u^{k+1}.$$

It is easy to see that $\mathcal{Y}_1 \to E \cap U_j$ is a trivial family whose fiber is the normalization of the central fiber $\mathcal{C}_0$. Further, $\mathcal{Y}_2$ is a family of divisors in the weighted projective space $\mathbb{P}(2, 2, k+1)$ given by the quasi-homogeneous (in variables x, u, y) equation

$$\{y^2 = x^{k+1} + c_{k-1}^2 u^2 x^{n-1} + \cdots + u^j x^{k+1-j} + \cdots + c_0^{k+1} u^{k+1}\} \subset \mathbb{P}(2, 2, k+1) \times (E \cap U_j).$$

In particular, $\mathcal{Y}_2$ admits a $2:1$ morphism to $\mathbb{P}(2,2)\times(E\cap U_j)\simeq\mathbb{P}^1\times(E\cap U_j)$ given by $(x,u,y)\mapsto(x,u)$. Since the ramification divisor has degree $k+1$ when k is odd and $k+2$ when k is even and has no points of multiplicity $k+1$, we conclude that $\mathcal{Y}_2\to(E\cap U_j)$ is a family of curves in $\mathcal{H}_k[k-1]$. Moreover, $\mathcal{Y}_2$ is attached to $\mathcal{Y}_1$ along the locus $(u=0)\cap\mathcal{Y}_2$ in $\mathbb{P}(2,2,k+1)\times(E\cap U_j)$. Since the equation $y^2=x^{k+1}$ defines two points in $\mathbb{P}(2,k+1)$ when k is odd and a single point when k is even, we conclude that, for every $t\in f^{-1}(0)$, the curve $(\mathcal{Y}_2)_t$ is attached to $(\mathcal{Y}_1)_t$ along a ramification point when k is even and along two conjugate points if k is odd. ☐

Recently, Casalaina-Martin and Laza have described an explicit alteration[2] sufficient to regularize the moduli map in the case when $\hat{\mathcal{O}}_{C,p}$ is an ADE singularity [CML10, Main Theorem]. We briefly describe their approach:

First, one passes to the Weyl cover of $\mathrm{Def}(C)$ (see [CML10, Section 2] and references therein) – a finite base extension after which the discriminant divisor $\Delta_C\subset\mathrm{Def}(C)$ becomes an arrangement of hyperplanes. After the wonderful blow-up of [DCP95] the discriminant becomes a simple normal crossing divisor. An application of the extension theorem of de Jong-Oort [dJO97, Theorem 5.1] (see also [Cau09, Theorem 1.2]) now gives a morphism to the coarse moduli space $\overline{M}_g$. Finally, one verifies that the family of stable curves away from the discriminant extends to a family over the generic point of every irreducible component of the discriminant, except for the components corresponding to A_{2k} singularities. A further finite base change is required to obtain the extension of the morphism to the moduli stack $\overline{\mathcal{M}}_g$. We refer to [CML10] for more details.

Remark 8.3 (Stable reduction over higher-dimensional bases)**.** Note that given an irreducible scheme T of any dimension and a stable curve over the generic point of T, there is always an alteration of T after which (the pullback of) the stable curve over the generic point extends over the whole base. Indeed, since $\overline{\mathcal{M}}_g$ is a proper stack over $\mathbb{Z}$ with a finite diagonal, by [EHKV01, Theorem 2.7] there exists a finite surjective morphism $V\to\overline{\mathcal{M}}_g$, with V a scheme. The rational map $T\dashrightarrow\overline{\mathcal{M}}_g$ lifts[3] to a rational map $T'\dashrightarrow V$, where T' maps finitely to T. Since V is proper, there exists a proper birational morphism $T''\to T'$ after which this rational map extends. The composition $T''\to T'\to T$ is the requisite alteration.

We now explain where the necessity in working with Deligne-Mumford stacks originates. This is accomplished by the following proposition.

Proposition 8.4. *There exists a curve C with a unique isolated A_{2k} singularity such that the indeterminacy of the moduli map* $\mathrm{Def}(C)\dashrightarrow\overline{\mathcal{M}}_{p_a(C)}$ *cannot be resolved by any proper birational modification of* $\mathrm{Def}(C)$ *which restricts to an isomorphism over* $\mathrm{Def}(C)\smallsetminus\Delta_C$*. Furthermore, the indeterminacy cannot be resolved by any proper modification of* $\mathrm{Def}(C)$ *which is isomorphic to the Weyl cover of* $\mathrm{Def}(C)$ *over* $\mathrm{Def}(C)\smallsetminus\Delta_C$*.*

Proof. To prove the first part it suffices to exhibit C and a smoothing $f\colon\mathcal{C}\to(T,0)$ of C such that the stable curve $\mathcal{C}\times_T(T\smallsetminus 0)\to T\smallsetminus 0$ does not extend to a stable curve over T.

To begin, consider a family of stable $(4k+2)$-pointed rational curves $\mathcal{Y}\to(T,0)$ (over a spectrum of a DVR) with the central fiber $Y_0=E_1\cup E_2$ – a nodal union of two rational

[2]An alteration is a composition of a proper birational morphism with a finite morphism [dJ96].

[3]The lifting exists due to the extension property of coherent sheafs.

curves each marked by exactly $2k+1$ sections. Assume that the total space of $\mathcal{Y}$ is smooth. Denote by Σ the union of all $4k+2$ sections. Next, let $T' \to T$ be a finite base extension of degree 2, ramified over 0. The fiber product $\mathcal{Y}' = \mathcal{Y} \times_T T'$ has a (surface) singularity of type A_1 lying over the node of Y_0. Make an ordinary blow-up with the center at this singularity, and denote by F the exceptional divisor. By, e.g., Lemma 3.10, the divisor $\Sigma + F$ is divisible by 2 in the Picard group of the blown-up surface. We can now construct a cyclic 2-cover branched over $\Sigma + F$. The resulting cover is smooth and contains a (-1)-curve – the preimage of F. Consider the Stein factorization of the morphism from the said cyclic cover to $\mathcal{Y}'$. The resulting Stein morphism blows-down the (-1)-curve, and the resulting surface $\mathcal{X}'$ admits a finite, degree 2 morphism to Y'.

Note that, by construction, the central fiber of $\mathcal{X}'$ is a nodal union $X_1 \cup X_2$ of two hyperelliptic genus k curves. The line bundle $\omega_{\mathcal{X}'/T'}((2k-1)X_1)$ is relatively base-point-free: Clearly, it has no base points away from X_1, and by considering the divisor $(4k-2)\tau$, where τ is a Weierstrass section of $\mathcal{X}'/T'$ disjoint from X_1, we conclude that it has no base points along X_1. Since $\omega_{\mathcal{X}'/T'}((2k-1)X_1)|_{X_1} \simeq \mathcal{O}_{X_1}$, it follows that $\omega_{\mathcal{X}'/T'}((2k-1)X_1)$ defines a T'-morphism $\mathcal{X}' \to \mathcal{X}''$ contracting X_1 to a point. A simple application of the theorem on formal functions shows that the image of X_1 is an A_{2k} singularity of the central fiber X''_0.

Finally, we take $C := X''_0$. By the construction, we have that $\mathcal{X}''_{|T' \smallsetminus 0}$ is a base extension of a smooth family of genus $2k$ curves over $T \smallsetminus 0$ and so the morphism $T' \smallsetminus 0 \to \overline{\mathcal{M}}_{2k}$ factors through $T \smallsetminus 0$. However, since $\mathcal{X}''$ is smooth, $\mathcal{X}'' \to T'$ cannot be a base extension of a stable family of genus $2k$ curves over T. It follows that the morphism $T \smallsetminus 0 \to \overline{\mathcal{M}}_{2k}$ does not extend to T. This establishes the first part of the proposition.

The second part follows analogously due to the fact that, by the above construction, the morphism from T to $\mathrm{Def}(C)$ factors through the Weyl cover of $\mathrm{Def}(C)$. □

Remark 8.5. We remark that Proposition 8.4 strengthens the non-existence part of [CML10, Theorem 6.1], which is proven there using monodromy considerations.

8.2. **Adjacencies of D singularities.** Section 7 establishes a connection between deformations of singularities of type D and deformations of pointed quasi-admissible covers. Using it, we recover a well-known result on adjacencies of D singularities [Arn76].

Proposition 8.6. *A collection of singularities appearing on a small deformation of a curve singularity of type D_n is necessarily of the form $\{D_{k_0}, A_{k_1}, \dots, A_{k_r}\}$, where $k_0 + \sum_{i=1}^{r}(k_i + 1) \leq n$. Here, $r \geq 0$ and $k_0 \geq 0$, with the further convention that D_0 and D_1 stand for smooth points, D_2 stands for two A_1 singularities, and $D_3 = A_3$.*

Proof. Given what has been done so far, the proof becomes an observation: By Theorem 7.1, a quasi-admissible cover with a D_n singularity corresponds to a cover where the marked point χ lies over a point of multiplicity n of the branch divisor. By Proposition 4.13, all deformations of D_n are realized as deformations of the cover. In a deformation of the cover, the branch point of multiplicity n deforms into points of multiplicities $k_0, k_1+1, \dots, k_r+1$, where the marked point χ lies over the point of multiplicity $k_0 \geq 0$. The singularities of the deformed cover are precisely $D_{k_0}, A_{k_1}, \dots, A_{k_r}$. □

References

[ACV03] Dan Abramovich, Alessio Corti, and Angelo Vistoli. Twisted bundles and admissible covers. *Comm. Algebra*, 31(8):3547–3618, 2003. Special issue in honor of Steven L. Kleiman. 3, 12, 17

[AGLV98] V. I. Arnol′d, V. V. Goryunov, O. V. Lyashko, and V. A. Vasil′ev. *Singularity theory. I.* Springer-Verlag, Berlin, 1998. 6

[Arn75] V. I. Arnol′d. Critical points of smooth functions, and their normal forms. *Uspehi Mat. Nauk*, 30(5(185)):3–65, 1975. 6

[Arn76] V. I. Arnol′d. Local normal forms of functions. *Invent. Math.*, 35:87–109, 1976. 6, 34

[ASvdW10] Jarod Alper, David Smyth, and Fred van der Wyck. Weakly proper moduli stacks of curves. `arXiv:1012.0538 [math.AG]`, 2010. 5

[AV02] Dan Abramovich and Angelo Vistoli. Compactifying the space of stable maps. *J. Amer. Math. Soc.*, 15(1):27–7, 2002. 12, 16, 22

[AV04] Alessandro Arsie and Angelo Vistoli. Stacks of cyclic covers of projective spaces. *Compos. Math.*, 140(3):647–666, 2004. 5, 20

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1990. 14

[Cad07] Charles Cadman. Using stacks to impose tangency conditions on curves. *Amer. J. Math.*, 129(2):405–427, 2007. 15

[Cau09] Sabin Cautis. The abelian monodromy extension property for families of curves. *Math. Ann.*, 344(3):717–747, 2009. 33

[CML10] Sebastian Casalaina-Martin and Radu Laza. Simultaneous semi-stable reduction for curves with ADE singularities. `arXiv:1007.0265 [math.AG]`, 2010. 9, 33, 34

[Con] Brian Conrad. The Keel–Mori theorem via stacks. Available at http://math.stanford.edu/ conrad/papers/coarsespace.pdf. 10, 15

[DCP95] C. De Concini and C. Procesi. Wonderful models of subspace arrangements. *Selecta Math. (N.S.)*, 1(3):459–494, 1995. 33

[dJ96] A. J. de Jong. Smoothness, semi-stability and alterations. *Inst. Hautes Études Sci. Publ. Math.*, (83):51–93, 1996. 33

[dJO97] A. J. de Jong and F. Oort. On extending families of curves. *J. Algebraic Geom.*, 6(3):545–562, 1997. 33

[DM69] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, (36):75–109, 1969. 2, 7

[EHKV01] Dan Edidin, Brendan Hassett, Andrew Kresch, and Angelo Vistoli. Brauer groups and quotient stacks. *Amer. J. Math.*, 123(4):761–777, 2001. 33

[Fed07] Maksym Fedorchuk. Linear sections of the Severi variety and moduli of curves. Available at `arXiv:0710.1623 [math.AG]`, 2007. 32

[Fed11] Maksym Fedorchuk. Moduli of weighted pointed stable curves and log canonical models of $\overline{\mathcal{M}}_{g,n}$. *Math. Res. Lett.*, 18(4):663–675, 2011. 23, 25, 27

[Ful05] Damiano Fulghesu. *On the Chow ring of the moduli stack of rational nodal curves*. PhD thesis, Scuola Normale Superiore, Pisa, 2005. 10

[Has00] Brendan Hassett. Local stable reduction of plane curve singularities. *J. Reine Angew. Math.*, 520:169–194, 2000. 7

[Has03] Brendan Hassett. Moduli spaces of weighted pointed stable curves. *Adv. Math.*, 173(2):316–352, 2003. 9, 10, 11, 23, 24, 27, 28

[HH08] Brendan Hassett and Donghoon Hyeon. Log minimal model program for the moduli space of curves: the first flip. Submitted. Available at `arXiv:0806.3444v1 [math.AG]`, 2008. 2

[HH09] Brendan Hassett and Donghoon Hyeon. Log canonical models for the moduli space of curves: the first divisorial contraction. *Trans. Amer. Math. Soc.*, 361(8):4471–4489, 2009. 2, 26, 27

[HL10] Donghoon Hyeon and Yongnman Lee. Log minimal model program for the moduli space of stable curves of genus three. *Math. Res. Lett.*, 17(4):625–636, 2010. 2

[HM82] Joe Harris and David Mumford. On the Kodaira dimension of the moduli space of curves. *Invent. Math.*, 67(1):23–88, 1982. With an appendix by William Fulton. 3, 17

[KM97] Seán Keel and Shigefumi Mori. Quotients by groupoids. *Ann. of Math. (2)*, 145(1):193–213, 1997. 10

[KM04] T. Khirsh and B. Martin. Deformations with section: cotangent cohomology, flatness conditions, and modular subgerms. *Sovrem. Mat. Prilozh.*, (15, Teor. Funkts.):52–70, 2004. 30

[Kol97] János Kollár. Singularities of pairs. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 221–287. Amer. Math. Soc., Providence, RI, 1997. 6

[LMB00] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 2000. 15

[MvS01] D. Mond and D. van Straten. The structure of the discriminant of some space-curve singularities. *Q. J. Math.*, 52(3):355–365, 2001. 30

[Pin74] Henry C. Pinkham. *Deformations of algebraic varieties with G_m action*. Société Mathématique de France, Paris, 1974. Astérisque, No. 20. 5

[Ryd10] D. Rydh. Representability of Hilbert schemes and Hilbert stacks of points. To appear in Comm. Alg (2010), `arXiv:0802.3807 [math.AG]`, 2010. 10

[Ser06] Edoardo Sernesi. *Deformations of algebraic schemes*, volume 334 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 2006. 6

[Tju69] G. N. Tjurina. Locally semi-universal flat deformations of isolated singularities of complex spaces. *Izv. Akad. Nauk SSSR Ser. Mat.*, 33:1026–1058, 1969. 6

Department of Mathematics, Columbia University, 2990 Broadway, New York, NY 10027
E-mail address: `mfedorch@math.columbia.edu`