STABILITY OF ASSOCIATED FORMS

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Abstract. We show that the associated form, or, equivalently, a Macaulay inverse system, of an Artinian complete intersection of type $(d, \dots, d)$ is polystable. As an application, we obtain an invariant-theoretic variant of the Mather-Yau theorem for homogeneous hypersurface singularities.

Contents

1. Introduction

In this paper, we establish the GIT polystability of Macaulay inverse systems for Gorenstein Artin algebras given by balanced complete intersections. This leads to a purely invariant-theoretic solution to the problem of deciding when two such algebras are isomorphic. An important example of a balanced complete intersection is the Milnor algebra of an isolated homogeneous hypersurface singularity and so, as an application of our polystability result, we obtain an algebraic variant of the Mather-Yau theorem for such singularities over an arbitrary field of characteristic zero.

We will now explain our approach. Recall that for two homogeneous forms $f(x_1, \dots, x_n)$ and $g(x_1, \dots, x_n)$ over a field k, the problem of determining whether one can be obtained from the other by a linear change of variables can often be solved by a purely algebraic method offered by Geometric Invariant Theory (GIT). Namely, if $\deg f = \deg g = d$, then the question can be rephrased as whether we have the equality of the orbits

$$\mathrm{GL}(n) \cdot f = \mathrm{GL}(n) \cdot g$$

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under the natural action of $\mathrm{GL}(n)$ on $\mathrm{Sym}^d V$, where V is the standard representation of $\mathrm{GL}(n)$. When f and g are polystable in the sense of GIT, their orbits in $\mathrm{Sym}^d V$ can be distinguished using invariants. Namely, the two orbits are distinct if and only if there exists a homogeneous $\mathrm{SL}(n)$-invariant $\mathfrak{I}$ on $\mathrm{Sym}^d V$, that is, an $\mathrm{SL}(n)$-invariant homogeneous element of $\mathrm{Sym}\big(\mathrm{Sym}^d V\big)^\vee$, such that $\mathfrak{I}(f) = 0$ and $\mathfrak{I}(g) \neq 0$. Furthermore, the Gordan-Hilbert theorem implies that we can find finitely many homogeneous $\mathrm{SL}(n)$-invariants $\mathfrak{I}_1, \dots, \mathfrak{I}_N$ of equal degrees such that for polystable f and g, we have

$$\mathrm{GL}(n) \cdot f = \mathrm{GL}(n) \cdot g \quad \Longleftrightarrow \quad [\mathfrak{I}_1(f) : \cdots : \mathfrak{I}_N(f)] = [\mathfrak{I}_1(g) : \cdots : \mathfrak{I}_N(g)]. \tag{1.1}$$

One can consider a generalization of the above question and ask when two m-dimensional linear systems $\langle f_1, \dots, f_m\rangle$ and $\langle g_1, \dots, g_m\rangle$ of degree d forms are related by a linear change of variables. This again can be phrased in terms of a GIT problem, this time given by the action of $\mathrm{SL}(n)$ on $\mathrm{Grass}(m, \mathrm{Sym}^d V)$ (or the $\mathrm{SL}(n)$-action on the affine cone over the Grassmannian in its Plücker embedding). A priori, to distinguish orbits of this action, one needs to understand polynomial $\mathrm{SL}(n)$-invariants on $\wedge^m \mathrm{Sym}^d V$. One of the main results of this paper is that for $m = n$, i.e., when the number of forms is equal to the number of variables, the problem of distinguishing $\mathrm{SL}(n)$-orbits in $\mathrm{Grass}(n, \mathrm{Sym}^d V)$ can often be reduced to that of distinguishing $\mathrm{SL}(n)$-orbits of degree $n(d-1)$ forms in n (dual!) variables.

The reason behind this simplification is that a generic n-dimensional subspace $U \subset \mathrm{Sym}^d V$ is spanned by a regular sequence $g_1, \dots, g_n$. The algebra

$$\mathcal{A} := k[x_1, \dots, x_n]/(g_1, \dots, g_n)$$

is then a graded local Gorenstein Artin algebra of socle degree $n(d-1)$. The homogeneous Macaulay inverse system of this algebra is then an element of $\mathbb{P}\,\mathrm{Sym}^{n(d-1)} V^\vee$, which we call the associated form of U. A classical theorem of Macaulay says that the associated form morphism $\mathbf{A}$ sending U to its associated form is injective (see Theorem 2.2). Alper and Isaev, who initiated a systematic study of this morphism, showed that $\mathbf{A}$ is a locally closed immersion, and conjectured that $\mathbf{A}$ preserves GIT semistability and that the induced morphism on the GIT quotients is also a locally closed immersion; we refer the reader to [2] for details, for the motivation behind these conjectures, and for a proof in the case of binary forms. In [12], the first author proved that $\mathbf{A}$ preserves GIT semistability. Here we show:

Theorem 1.2 (Theorem 2.6)**.** *Assume* $\mathrm{char}(k) = 0$. *Suppose that an element* $U \in \mathrm{Grass}(n, \mathrm{Sym}^d V)$ *is spanned by a regular sequence and is polystable. Then* $\mathbf{A}(U)$ *is polystable.*

Consequently, injectivity is preserved on the level of GIT quotients, just as Alper and Isaev conjectured.

As an application of Theorem 1.2, we obtain an invariant-theoretic variant of the Mather-Yau theorem for isolated homogeneous hypersurface singularities. The original version of this theorem, proved in [23], states that an isolated hypersurface singularity in $\mathbb{C}^n$ is determined, up to biholomorphism, by n and the isomorphism class of its moduli (Tjurina) algebra. The theorem was extended to the case of non-isolated hypersurface singularities in [13, Theorem 2.26]. Further, in [14, Proposition 2.1] it was established for arbitrary algebraically closed fields of characteristic 0. Finally, for singularities over algebraically closed fields of arbitrary characteristic, an analogue of the theorem was proved in [14, Theorem 2.2]. For more details on the history of the Mather-Yau theorem we refer the reader to [14].

The Mather-Yau theorem is non-trivial even for homogeneous singularities and raises a natural question of how exactly a singularity is encoded by the corresponding algebra (see [5]). If $f(x_1, \dots, x_n) = 0$ is such a singularity, defined by a form of degree $d+1$, then its moduli algebra coincides with its Milnor algebra M_f, which has an associated form $A(f) \in \operatorname{Sym}^{n(d-1)} V^\vee$, first studied in [1]. Our polystability result implies that two forms $f, g \in \operatorname{Sym}^{d+1} V$ define isomorphic isolated hypersurface singularities (i.e., the completions of the local rings of the hypersurfaces $\{f = 0\}$ and $\{g = 0\}$ at the origin are isomorphic over the algebraic closure of the field) if and only if their associated forms $A(f)$ and $A(g)$ map to the same point in the GIT quotient $\mathbb{P}\operatorname{Sym}^{n(d-1)} V^\vee /\!/ \operatorname{SL}(n)$, something that can be detected by finitely many homogeneous $\operatorname{SL}(n)$-invariants just as in (1.1). Since the associated form $A(f)$ is computable from the Milnor algebra alone, we obtain a purely algebraic, and in principle algorithmic, way of deciding when two isolated homogeneous hypersurface singularities are isomorphic based solely on their Milnor algebras:

Theorem 1.3 (Theorem 5.2)**.** *There exists a finite collection of homogeneous* $\operatorname{SL}(n)$-*invariants* $\mathfrak{I}_1, \dots, \mathfrak{I}_N$ *on* $\operatorname{Sym}^{n(d-1)} V^\vee$ *of equal degrees, defined over* k*, such that for any two forms* $f, g \in \operatorname{Sym}^{d+1} V$ *defining isolated singularities, the two singularities are isomorphic if and only if*

$$[\mathfrak{I}_1(A(f)) : \cdots : \mathfrak{I}_N(A(f))] = [\mathfrak{I}_1(A(g)) : \cdots : \mathfrak{I}_N(A(g))].$$

Notation and conventions. We work over a field k of characteristic 0 (not necessarily algebraically closed). The dual of a k-vector space will be denoted by $^\vee$. Fix $n \geq 1$ and let V be an n-dimensional vector space over k. Let $S := \operatorname{Sym} V$ be the symmetric algebra on V with the standard grading.

We briefly recall some basic notions of GIT utilized in this paper. Our main reference for GIT is [24], but the reader is also referred, e.g., to [20, Chapter 9] for a more elementary exposition that uses modern terminology. Suppose W is an algebraic representation of a reductive group G. Then $x \in W$ is *semistable* if $0 \notin \overline{G \cdot x}$ and *polystable* if $G \cdot x$ is closed. Similarly, for $\bar{x} \in \mathbb{P}W$, we say that $\bar{x}$ is semistable (resp., polystable) if some (equivalently, any) lift x of $\bar{x}$ to W is semistable (resp., polystable). The locus of semistable points in $\mathbb{P}W$ is open and is denoted by $\mathbb{P}W^{ss}$. More generally, if $X \subset \mathbb{P}W$ is a G-invariant projective closed subscheme, then one defines the locus of semistable points in X as $X^{ss} := X \cap \mathbb{P}W^{ss}$. The orbits of polystable k-points in X^{ss} are in bijection with the k-points of the projective GIT quotient

$$X^{ss}/\!/G := \operatorname{Proj} \bigoplus_{m\geq 0} \mathrm{H}^0\big(X, \mathcal{O}_X(m)\big)^G.$$

In particular, polystable orbits in $\mathbb{P}W^{ss}$ (and more generally in X^{ss}) are distinguished by G-invariant forms on $\mathbb{P}W$. It will be crucial for us that in the case of a perfect field, semistability and polystability is determined by the standard Hilbert-Mumford numerical criterion; see [19] for more details.

Since the definition of the associated form $A(f)$ requires a large enough characteristic (at the very least, we need $\operatorname{char}(k) \nmid \deg(f)$ in order for the partial derivatives of f to form a regular sequence), and our proof of polystability relies on characteristic 0 results, our Theorems 1.2 and 1.3 require $\operatorname{char}(k) = 0$. The reader can verify that as long as $A(f)$ is defined and the field is perfect, our proof of the semistability of $A(f)$ goes through. At the moment, we are not aware of any counterexamples to the polystability statement of Theorem 1.2 for fields of (sufficiently large) positive characteristic.

Roadmap of the paper. In Section 2, we introduce the main actors of this work, the balanced complete intersection algebras and their associated forms, and state our principal result (Theorem 2.6). In Section 3, we prove a key technical commutative algebra proposition. In Section 4, we prove Theorem 2.6. Finally, in Section 5, we give applications of our preservation of polystability result, the main of which is an invariant-theoretic variant of the Mather-Yau theorem.

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2. Associated forms of complete intersections

2.1. **Gorenstein Artin algebras and Macaulay inverse systems.** We briefly recall basics of the theory of Macaulay inverse systems of graded Gorenstein Artin algebras necessary to state our main result, but the reader is encouraged to consult [15] for a more comprehensive discussion.

Recall that a homogeneous ideal $I \subset S$ is *Gorenstein* if $\mathcal{A} := S/I$ is a *Gorenstein Artin k-algebra*, meaning that $\dim_k \mathcal{A} < \infty$ and $\dim_k \operatorname{Soc}(\mathcal{A}) = 1$. Here, $\operatorname{Soc}(\mathcal{A})$ is the annihilator of the unique maximal ideal $\mathfrak{m}_{\mathcal{A}}$ of $\mathcal{A}$. We endow $\mathcal{A}$ with the standard grading coming from S. Then

$$\mathcal{A} = \bigoplus_{d=0}^{\nu} \mathcal{A}_d,$$

where ν is the *socle degree* of $\mathcal{A}$, and $\mathrm{Soc}(\mathcal{A}) = \mathcal{A}_\nu$. The surjection $H_\nu \colon S_\nu \twoheadrightarrow \mathcal{A}_\nu$ is called the ν^{th} *Hilbert point* of $\mathcal{A}$, which we regard as a point in $\mathbb{P}S_\nu^\vee$. As we will see shortly, it is dual to the homogeneous Macaulay inverse system of $\mathcal{A}$.

We can regard $S = \mathrm{Sym}\, V$ as a ring of polynomial differential operators on a 'dual ring' $D := \mathrm{Sym}\, V^\vee$ as follows. Let $x_1, \dots, x_n$ be a basis of V and $z_1, \dots, z_n$ be the dual basis of $V^\vee$. Then we have an *apolarity action* of S on D

$$\circ \colon S \times D \to D$$

given by differentiation

$$g(x_1, \dots, x_n) \circ f(z_1, \dots, z_n) := g(\partial/\partial z_1, \dots, \partial/\partial z_n) f(z_1, \dots, z_n).$$

Since $\mathrm{char}(k) = 0$, the restricted pairing $S_d \times D_d \to k$ is perfect and so defines an isomorphism

$$D_d \simeq S_d^\vee \tag{2.1}$$

(see [15, Appendix A, Example A.5] and [18, Proposition 2.8] for more details).

Recall now the following classical result, whose modern exposition can be found in [15, Lemmas 2.12 and 2.14] or [10, Exercise 21.7].

Theorem 2.2 (Macaulay's theorem [22, Chapter IV])**.** *For every non-zero $f \in D_\nu$, the homogeneous ideal*

$$f^\perp := \{g \in S \mid g \circ f = 0\}$$

is such that $S/f^\perp$ is a Gorenstein Artin k-algebra of socle degree ν. Conversely, for every homogeneous Gorenstein ideal $I \subset S$ such that S/I has socle degree ν, there exists $f \in D_\nu$ such that $I = f^\perp$. Moreover, $f_1^\perp = f_2^\perp$ if and only if f_1 and f_2 are scalar multiples of each other.

Definition 2.3. If $I \subset S$ is a Gorenstein ideal and ν is the socle degree of the algebra $\mathcal{A} = S/I$, then a *(homogeneous) Macaulay inverse system* of $\mathcal{A}$ is an element $f \in D_\nu$, given by the above theorem, such that $I = f^\perp$.

Clearly, all Macaulay inverse systems are mutually proportional and the line $\langle f \rangle \in \mathbb{P}D_\nu$ maps to the ν^{th} Hilbert point $H_\nu \in \mathbb{P}S_\nu^\vee$ of $\mathcal{A}$ under isomorphism (2.1) for $d = \nu$. This leads to the following useful consequence of Theorem 2.2:

Corollary 2.4 (cf. [15, Lemmas 2.15 and 2.17])**.** *Let I and J be homogeneous ideals in S such that S/I and S/J are Gorenstein Artin k-algebras of socle degree ν. Then:*

1. *$I_d = \{g \in S_d \mid hg \in I_\nu$ for all $h \in S_{\nu-d}\}$, for $1 \le d \le \nu$.*
2. *$I = J$ if and only if $I_\nu = J_\nu$.*

Proof. (1) implies (2) and follows immediately from Macaulay's theorem by noting that for $\mathcal{A} = S/I$ the pairing $\mathcal{A}_d \times \mathcal{A}_{\nu-d} \to \mathcal{A}_\nu$ is perfect for every $d \le \nu$. ☐

For any $\omega \in S_\nu^\vee$ with $\ker \omega = I_\nu$, papers [8, 9] introduced an *associated form* of $\mathcal{A}$ as the element of D_ν given by the formula

$$f_{\mathcal{A},\omega} := \omega\big((x_1 z_1 + \cdots + x_n z_n)^\nu\big) \in k[z_1, \dots, z_n]_\nu. \tag{2.5}$$

Since $f_{\mathcal{A},\omega}^{\perp} = I$, these associated forms give explicit formulae for the Macaulay inverse systems of $\mathcal{A}$ (see [16] for more details).

2.2. **Koszul complex.** Suppose m is a positive integer. Recall that for $g_1, \dots, g_m \in S_d$, the Koszul complex $K_\bullet(g_1, \dots, g_m)$ is defined as follows. Let $e_1, \dots, e_m$ be the standard degree d generators of the graded free S-module $S(-d)^m$. Then $K_\bullet(g_1, \dots, g_m)$ is an $(m+1)$-term complex of graded free S-modules with

$$K_j(g_1, \dots, g_m) := \wedge^j S(-d)^m \text{ for } j = 1, \dots, m, \qquad K_0(g_1, \dots, g_m) := S,$$

and the differential $d_j \colon K_j(g_1, \dots, g_m) \to K_{j-1}(g_1, \dots, g_m)$ given by

$$d_j(e_{i_1} \wedge \dots \wedge e_{i_j}) := \sum_{r=1}^{j} (-1)^{k-1} g_{i_r} e_{i_1} \wedge \dots \wedge \widehat{e_{i_r}} \wedge \dots \wedge e_{i_j}.$$

Note that $\mathrm{H}_0(K_\bullet(g_1, \dots, g_m)) = S/(g_1, \dots, g_m)$. We will use without further comment basic results about Koszul complexes as developed in [10, Chapter 17].

2.3. **Balanced complete intersections and their associated forms.** Suppose $d \geq 2$ and $m \leq n$. Recall that elements $g_1, \dots, g_m \in S_d$ form a regular sequence in S if and only if any of the following equivalent conditions hold:

(1) $\operatorname{codim}(g_1, \dots, g_m) = m$.
(2) the Koszul complex $K_\bullet(g_1, \dots, g_m)$ is a minimal free resolution of $S/(g_1, \dots, g_m)$,

Moreover, if $n = m$, then the above conditions are also equivalent to each of

(3) the forms $g_1, \dots, g_n$ have no non-trivial common zero in $\bar{k}^n$.
(4) the resultant $\operatorname{Res}(g_1, \dots, g_n)$ is non-zero.

We now recall the definition of the associated form of a complete intersection as first given in [2]. To begin, if $g_1, \dots, g_n \in S_d$ form a regular sequence in S, then we call $I := (g_1, \dots, g_n)$ a *complete intersection ideal of type* $(d)^n$, or simply a *balanced complete intersection* if the degree d and the number of variables n are understood; here, "balanced" refers to the fact that $g_1, \dots, g_n$ have the same degree. In this case, we also call the algebra $\mathcal{A} := S/I$ a *complete intersection algebra of type* $(d)^n$, or a *balanced complete intersection*. A complete intersection algebra of type $(d)^n$ is a graded Gorenstein Artin k-algebra with Hilbert function

$$\sum_{j \geq 0} \dim_k(\mathcal{A}_j) t^j = \left(\frac{1 - t^d}{1 - t}\right)^n$$

and so has socle degree $n(d-1)$.

Let $H_{n(d-1)} \colon S_{n(d-1)} \to \mathcal{A}_{n(d-1)}$ be the $n(d-1)^{th}$ Hilbert point of $\mathcal{A}$. Denote by $\operatorname{Jac}(g_1, \dots, g_n)$ the Jacobian $n \times n$ matrix of $g_1, \dots, g_n$, whose $(ij)^{th}$ entry is $\partial g_i / \partial x_j$. Then $\mathcal{A}_{n(d-1)}$ is spanned by $H_{n(d-1)}(\det \operatorname{Jac}(g_1, \dots, g_n))$ (see [25, p. 187]), and so we can choose an isomorphism $\mathcal{A}_{n(d-1)} \simeq k$ that sends $H_{n(d-1)}(\det \operatorname{Jac}(g_1, \dots, g_n))$ to 1. Denote the resulting element of $S_{n(d-1)}^\vee = \operatorname{Hom}_k(S_{n(d-1)}, k)$ by ω. Then the form $f_{\mathcal{A},\omega} \in D_{n(d-1)}$ given by Equation (2.5) is called *the associated form of* $g_1, \dots, g_n$

and is denoted by $\mathbf{A}(g_1, \dots, g_n)$ (cf. [2]). The form $\mathbf{A}(g_1, \dots, g_n)$ is a homogeneous Macaulay inverse system of $\mathcal{A}$.

We let $\mathrm{Grass}(n, S_d)_{\mathrm{Res}}$ be the affine open subset of $\mathrm{Grass}(n, S_d)$ on which the resultant (considered as a section of the corresponding line bundle) does not vanish. Alper and Isaev defined the *associated form morphism*

$$\mathbf{A}\colon \mathrm{Grass}(n, S_d)_{\mathrm{Res}} \to \mathbb{P}D_{n(d-1)},$$

that sends a point $U \in \mathrm{Grass}(n, S_d)_{\mathrm{Res}}$ to the line spanned by $\mathbf{A}(g_1, \dots, g_n)$, where $g_1, \dots, g_n$ is any basis of U (see [2, Section 2]). By [2, Lemma 2.7], the morphism $\mathbf{A}$ is $\mathrm{SL}(n)$-equivariant. The preservation of GIT polystability by $\mathbf{A}$ is the main object of study in this paper. Our main result is Theorem 1.2, which we restate as follows:

Theorem 2.6. *Suppose $U \in \mathrm{Grass}(n, S_d)_{\mathrm{Res}}$ is polystable. Then $\mathbf{A}(U)$ is polystable.*

While proving Theorem 2.6, we also simplify the proof of the semistability of associated forms, first obtained by the first author in [12, Theorem 1.2]. We refer the reader to Theorem 4.6 for a more technical version of Theorem 2.6 that gives a necessary and sufficient condition for $\mathbf{A}(U)$ to be stable when k is algebraically closed.

2.4. **Balanced complete intersections and decomposability.** Among all codimension n ideals of S generated in degree d, the balanced complete intersections are distinguished using the following simple, but important lemma:

Lemma 2.7. *Suppose $J \subset S$ is a codimension n homogeneous ideal generated by J_d. Then either J is a balanced complete intersection or $(S/J)_{n(d-1)} = 0$.*

Proof. Since J has codimension n in S, there exist $r_1, \dots, r_n \in J_d$ that form a regular sequence. Set $Y := (r_1, \dots, r_n) \subset J$. Then S/Y is a balanced complete intersection algebra with socle in degree $n(d-1)$. Let $H_{n(d-1)}$ be the $n(d-1)^{th}$ Hilbert point of S/Y. We have two possibilities:

(1) either $H_{n(d-1)}((s)_{n(d-1)}) = 0$ for every $s \in J_d$, in which case $J = Y$ by Corollary 2.4,
(2) or there exists $s \in J_d$ such that $H_{n(d-1)}((s)_{n(d-1)}) \neq 0$.

In the latter case, $(s)_{n(d-1)} \not\subset Y_{n(d-1)}$, so that $J_{n(d-1)}$ strictly contains $Y_{n(d-1)}$. Since $Y_{n(d-1)}$ is already of codimension 1 in $S_{n(d-1)}$, we conclude that $J_{n(d-1)} = S_{n(d-1)}$ in this case. □

Definition 2.8. We say that $U \in \mathrm{Grass}(n, S_d)_{\mathrm{Res}}$ is *decomposable* if there is a choice of a basis $x_1, \dots, x_n$ of S_1, an integer $1 \le a \le n-1$, and a basis $g_1, \dots, g_n$ of U such that $g_{a+1}, \dots, g_n \in k[x_{a+1}, \dots, x_n]$. An element $U \in \mathrm{Grass}(n, S_d)_{\mathrm{Res}}$ that is not decomposable will be called *indecomposable.* For $U \in \mathrm{Grass}(n, S_d)_{\mathrm{Res}}$, we will also speak about the (in)decomposability of the balanced complete intersection ideal $I := (U) \subset S$ and the balanced complete intersection algebra S/I.

Decomposable complete intersections have a simple structure described by the following result:

Proposition 2.9. *Suppose* $U = \langle g_1, \dots, g_n \rangle \in \mathrm{Grass}(n, S_d)_{\mathrm{Res}}$ *is such that, for some* $1 \leq a \leq n-1$*, we have* $g_{a+1}, \dots, g_n \in k[x_{a+1}, \dots, x_n]_d$*. Then* $g_{a+1}, \dots, g_n$ *is a regular sequence in* $k[x_{a+1}, \dots, x_n]_d$ *and there exists a regular sequence* $g_1', \dots, g_a' \in k[x_1, \dots, x_a]_d$ *such that the closure of the* $\mathrm{SL}(n)$*-orbit of* $U \in \mathrm{Grass}(n, S_d)_{\mathrm{Res}}$ *contains* $\langle g_1', \dots, g_a', g_{a+1}, \dots, g_n \rangle$*.*

Proof. The first statement is clear. For the second, set $g_i' := g_i(x_1, \dots, x_a, 0, \dots, 0)$. Then $g_1', \dots, g_a'$ form a regular sequence in $k[x_1, \dots, x_a]$. Let μ be the 1-PS of $\mathrm{SL}(n)$ acting with weight $-(n-a)$ on $x_1, \dots, x_a$ and with weight a on $x_{a+1}, \dots, x_n$. Then

$$\lim_{t \to 0} \mu(t) \cdot U = \langle g_1', \dots, g_a', g_{a+1}, \dots, g_n \rangle \in \mathrm{Grass}(n, S_d)_{\mathrm{Res}}.$$

This finishes the proof. □

The following is immediate:

Corollary 2.10. *Suppose a decomposable* $U \in \mathrm{Grass}(n, S_d)_{\mathrm{Res}}$ *is polystable. Then there exists* $1 \leq a \leq n-1$ *and a basis* $x_1, \dots, x_n$ *of* S_1 *such that*

$$U = \langle g_1, \dots, g_a, g_{a+1}, \dots, g_n \rangle,$$

where $g_1, \dots, g_a$ *is a regular sequence in* $k[x_1, \dots, x_a]_d$ *and* $g_{a+1}, \dots, g_n$ *is a regular sequence in* $k[x_{a+1}, \dots, x_n]_d$*.*

The balanced complete intersections described by the previous corollary are called *direct sums*; we will see in Remark 4.9 that non-polystable balanced complete intersections are always decomposable, but are not direct sums of indecomposables. The associated forms of direct sums are computed as follows.

Lemma 2.11. *Suppose for* $1 \leq a \leq n-1$*, we have that* $g_1, \dots, g_a \in k[x_1, \dots, x_a]_d$ *is a regular sequence and* $g_{a+1}, \dots, g_n \in k[x_{a+1}, \dots, x_n]_d$ *is a regular sequence. Then*

$$\mathbf{A}(g_1, \dots, g_n) = \binom{n(d-1)}{a(d-1)} \mathbf{A}(g_1, \dots, g_a) \mathbf{A}(g_{a+1}, \dots, g_n),$$

where $\mathbf{A}(g_1, \dots, g_a)$ *is the associated form of* $g_1, \dots, g_a$ *in* $k[z_1, \dots, z_a]_{a(d-1)}$*, and* $\mathbf{A}(g_{a+1}, \dots, g_n)$ *is the associated form of* $g_{a+1}, \dots, g_a$ *in* $k[z_{a+1}, \dots, z_n]_{(n-a)(d-1)}$*.*

Proof. As an element of $S_{n(d-1)}^{\vee}$ under the isomorphism (2.1), $\mathbf{A}(g_1, \dots, g_n)$ is uniquely determined by the property that it vanishes on $(g_1, \dots, g_n)_{n(d-1)}$ and satisfies

$$\mathbf{A}(g_1, \dots, g_n)\big(\det \mathrm{Jac}(g_1, \dots, g_n)\big) = (n(d-1))!.$$

The claim follows using $\det \mathrm{Jac}(g_1, \dots, g_n) = \det \mathrm{Jac}(g_1, \dots, g_a) \det \mathrm{Jac}(g_{a+1}, \dots, g_n)$. □

3. Recognition criterion for decomposable balanced complete intersections

Decomposable balanced complete intersections play a crucial role in our inductive proof of polystability of associated forms. In this section, we obtain a criterion for a balanced complete intersection ideal to be decomposable based only on partial information about the ideal. Although technical, this result may be of independent interest; in fact, it has already been used by the first author to give a new criterion for forms defining smooth hypersurfaces to be of Sebastiani-Thom type [11]. We note that the results of this section are valid over an arbitrary field k, with no restriction on its characteristic.

Proposition 3.1. *Let $1 \leq b \leq n-1$ and suppose that $I \subset k[x_1, \dots, x_n]$ is a complete intersection ideal of type $(d)^n$ such that*

(A) *the homomorphic image of I in $k[x_1, \dots, x_n]/(x_{b+1}, \dots, x_n) \simeq k[x_1, \dots, x_b]$ is a balanced complete intersection ideal, equivalently,*

$$\dim_k\bigl(I_d \cap (x_{b+1}, \dots, x_n)\bigr) = n - b,$$

(B) $(x_{b+1}, \dots, x_n)^{(n-b)(d-1)+1} \subset I$.

Then there are $n-b$ linearly independent elements $g_{b+1}, \dots, g_n \in I_d \cap k[x_{b+1}, \dots, x_n]$; in particular, I is decomposable.

Remark 3.2. Note that conditions (A) and (B) are necessary for the conclusion to hold. Indeed, if $g_{b+1}, \dots, g_n \in I_d \cap k[x_{b+1}, \dots, x_n]$ is a regular sequence, then $I_d \cap (x_{b+1}, \dots, x_n) = (g_{b+1}, \dots, g_n)$, and $(g_{b+1}, \dots, g_n)$ is a balanced complete intersection ideal in $k[x_{b+1}, \dots, x_n]$ and so contains $(x_{b+1}, \dots, x_n)^{(n-b)(d-1)+1}$. The difficulty lies in verifying the sufficiency of these conditions.

Proof of Proposition 3.1. Note that for $b = n-1$, Proposition 3.1 is obvious, because in this case $x_n^d \in I_d$ by (B). So we assume that $b \leq n-2$ in what follows.

Using condition (A) we can find a basis $g_1, \dots, g_n$ of I_d such that $g_{b+1}, \dots, g_n \in (x_{b+1}, \dots, x_n)$. We will prove that in fact $g_{b+1}, \dots, g_n \in k[x_{b+1}, \dots, x_n]$. We separate our argument into two parts, given by two key Lemmas 3.3 and 3.11.

Lemma 3.3. *Suppose $I = (g_1, \dots, g_n)$ is a complete intersection ideal of type $(d)^n$ such that $g_{b+1}, \dots, g_n \in (x_{b+1}, \dots, x_n)$ and condition (B) of Proposition 3.1 holds. Then*

$$(x_{b+1}, \dots, x_n)^{(n-b)(d-1)+1} \subset (g_{b+1}, \dots, g_n).$$

Remark 3.4. The idea behind our proof of this lemma is to understand all the syzygy modules of the ideals $(x_{b+1}, \dots, x_n)^{(n-b)(d-1)+1}$ and $(g_1, \dots, g_n)$. Comparing syzygies of a certain order then gives the requisite statement. We encourage the reader to keep in mind the first non-trivial case given by a regular sequence $g_1, g_2, g_3 \in k[x_1, x_2, x_3]_d$ such that $g_2, g_3 \in (x_2, x_3)$ and $(x_2, x_3)^{2d-1} \subset (g_1, g_2, g_3)$. The lemma asserts in this case that in fact $(x_2, x_3)^{2d-1} \subset (g_2, g_3)$.

Proof of Lemma 3.3. Set $N := (n-b)(d-1)+1$, and let $J := (x_{b+1},\dots,x_n)^N$. Let $\tilde{J} := J \cap k[x_{b+1},\dots,x_n]$ and $R := k[x_{b+1},\dots,x_n]$. Then $\tilde{J}$ is the N^{th} power of the irrelevant ideal in R and so has regularity N, for example, by the computation of its local cohomology (cf. [4, Lemma 1.7]). Hence $\tilde{J}$ has a *linear* minimal free resolution as an R-module. In fact, as explained in [6, pp. 269-270], an explicit minimal free resolution of $\tilde{J}$ was constructed by Buchsbaum and Rim using the Eagon-Northcott complex [7]. It follows that the minimal free resolution of $\tilde{J}$ has the following form:

$$0 \to R^{\ell_{n-b}}(-N-n+b+1) \to \cdots \to R^{\ell_2}(-N-1) \to R^{\ell_1}(-N) \to \tilde{J} \to 0. \tag{3.5}$$

Since S is a flat R-algebra, tensoring by S we obtain a minimal free resolution of J as an S-module:

$$0 \to S^{\ell_{n-b}}(-N-n+b+1) \to \cdots \to S^{\ell_2}(-N-1) \to S^{\ell_1}(-N) \to J \to 0.$$

Consider now the Koszul complex $K_\bullet(g_1,\dots,g_n)$, which gives a minimal free resolution of S/I; we keep the notation of §2.2. By our assumption, we have an inclusion $J \subset I$ gives rise to a map of complexes

$$\begin{array}{ccccccc} S^{\ell_{n-b}}(-N-n+b+1) & \longrightarrow & \cdots & \longrightarrow & S^{\ell_1}(-N) & \longrightarrow & S \\ \downarrow{\scriptstyle m_{n-b}} & & \downarrow & & \downarrow{\scriptstyle m_1} & & \downarrow{\scriptstyle m_0=\mathrm{Id}_S} \\ K_{n-b}(g_1,\dots,g_n) & \longrightarrow & \cdots & \longrightarrow & K_1(g_1,\dots,g_n) & \longrightarrow & S. \end{array} \tag{3.6}$$

Next, note that the Koszul complex $K_\bullet(g_1,\dots,g_n)$ contains $K_\bullet(g_{b+1},\dots,g_n)$ as a subcomplex. Let $Q_\bullet$ be the quotient complex. Then

$$Q_i := K_i(g_1,\dots,g_n)/K_i(g_{b+1},\dots,g_n)$$

is a free S-module for every $i \geq 1$, and from the long exact sequence in homology associated to the short exact sequence of complexes

$$0 \to K_\bullet(g_{b+1},\dots,g_n) \to K_\bullet(g_1,\dots,g_n) \to Q_\bullet \to 0,$$

we obtain that $Q_0 = 0$, $\mathrm{H}_1(Q_\bullet) = I/(g_{b+1},\dots,g_n)$, and $\mathrm{H}_i(Q_\bullet) = 0$ for $i > 1$.

Composing (3.6) with the quotient morphism, and replacing Q_0 by $I/(g_{b+1},\dots,g_n)$, we obtain a map of exact complexes

$$\begin{array}{ccccccc} S^{\ell_{n-b}}(-N-n+b+1) & \longrightarrow & \cdots & \longrightarrow & S^{\ell_1}(-N) & \longrightarrow & S \\ \downarrow{\scriptstyle \tilde{m}_{n-b}} & & \downarrow & & \downarrow{\scriptstyle \tilde{m}_1} & & \downarrow{\scriptstyle \tilde{m}_0} \\ Q_{n-b} & \longrightarrow & \cdots & \longrightarrow & Q_1 & \xrightarrow{d_1} & I/(g_{b+1},\dots,g_n). \end{array} \tag{3.7}$$

Note that $J \subset (g_{b+1},\dots,g_n)$ if and only if

$$\tilde{m}_0(J) = \mathrm{Im}(d_1 \circ \tilde{m}_1) = 0,$$

which is what we are going to prove. We begin with the following:

Claim 3.8. *$\tilde{m}_{n-b} = 0$, or, equivalently,* $\mathrm{Im}(m_{n-b}) \subset K_{n-b}(g_{b+1},\dots,g_n)$.

Proof. The key observation is that

$$-N - n + b + 1 = -(n-b)d.$$

Since $K_{n-b}(g_1, \dots, g_n) \simeq S^{\binom{n}{n-b}}\big(-(n-b)d\big)$, it follows that m_{n-b} in (3.6) is given by a matrix of scalars. Hence it suffices to prove that

$$\operatorname{Im}(m_{n-b} \otimes_S \bar{S}) \subset K_{n-b}(g_{b+1}, \dots, g_n) \otimes_S \bar{S},$$

where $\bar{S} := S/(x_{b+1}, \dots, x_n)$. Upon tensoring (3.6) with $\bar{S}$, all differentials in the top row become zero because (3.5) was a minimal resolution of an ideal in $k[x_{b+1}, \dots, x_n]$. It follows that $\operatorname{Im}(m_{n-b} \otimes_S \bar{S}) \subset \ker(d_{n-b} \otimes_S \bar{S})$. Since $\operatorname{Im}(m_{n-b})$ is generated in graded degree $(n-b)d$, we at last reduce to showing that $\ker(d_{n-b} \otimes_S \bar{S})_{(n-b)d} \subset (K_{n-b}(g_{b+1}, \dots, g_n) \otimes_S \bar{S})_{(n-b)d}$. Let $\bar{g}_i$ be the image of g_i in $\bar{S}$ for $i = 1, \dots, n$; we have $\bar{g}_{b+1} = \cdots = \bar{g}_n = 0$ by our assumption. Then

$$\begin{aligned} K_{n-b}(g_1, \dots, g_n) \otimes_S \bar{S} &\simeq K_{n-b}(\bar{g}_1, \dots \bar{g}_b, 0, \dots, 0) \\ &\simeq \textstyle\bigoplus_{j=0}^{n-b} K_j(\bar{g}_1, \dots, \bar{g}_b) \otimes_k K_{n-b-j}(0, \dots, 0). \end{aligned} \tag{3.9}$$

Moreover, the inclusion $K_{n-b}(g_{b+1}, \dots, g_n) \subset K_{n-b}(g_1, \dots, g_n)$ induces the isomorphism $K_{n-b}(g_{b+1}, \dots, g_n) \otimes_S \bar{S} \simeq K_0(\bar{g}_1, \dots, \bar{g}_b)_0 \otimes_k K_{n-b}(0, \dots, 0)_{(n-b)d}$. In (3.9), $d_{n-b} \otimes_S \bar{S}$ restricts to $d_j \otimes 1$ on the summand $K_j(\bar{g}_1, \dots, \bar{g}_b) \otimes_k K_{n-b-j}(0, \dots, 0)$. In graded degree $(n-b)d$, this restriction is simply

$$d_j \otimes 1 \colon K_j(\bar{g}_1, \dots, \bar{g}_b)_{dj} \otimes \wedge^{n-b-j} R(-d)^{n-b} \to K_j(\bar{g}_1, \dots, \bar{g}_b)_{dj-1} \otimes \wedge^{n-b-j} R(-d)^{n-b}.$$

This map is injective for $n - b \geq j \geq 1$ since the matrices defining the Koszul differentials in $K_\bullet(\bar{g}_1, \dots, \bar{g}_b)$ are of full rank, as $\bar{g}_1, \dots, \bar{g}_b$ are linearly independent over k. The claim follows. ∎

We now proceed to prove that $\tilde{m}_{n-b} = 0$ implies $d_1 \circ \tilde{m}_1 = 0$. To lighten notation, we let $T_\bullet$ be the exact complex given by the top row in diagram (3.7), so that $T_i := S^{\ell_i}(-N - i + 1)$ for $i = 1, \dots, n-b$ and $T_0 := S$.

Claim 3.10. *The map of complexes $\tilde{m} \colon T_\bullet \to Q_\bullet$ is null-homotopic in homological degree $i \leq n-b$. Namely, for $i = 0, \dots, n-b-1$, there exist maps $h_i \colon T_i \to Q_{i+1}$ such that*

$$\tilde{m}_i = d_{i+1} \circ h_i + h_{i-1} \circ d_i, \quad \textit{for every } i = 1, \dots, n-b-1.$$

Proof. It suffices to prove that the dual map $\tilde{m}^\vee \colon \operatorname{Hom}(Q_\bullet, S) \to \operatorname{Hom}(T_\bullet, S)$ is null-homotopic. Note that the top row of (3.7) is the resolution of the S-module S/J. Since $\operatorname{Ext}^j(S/J, S)$ vanishes for $j < \operatorname{codim} J = \dim S - \dim(S/J) = n - b$, the complex $\operatorname{Hom}(T_\bullet, S)$ gives a resolution of $\operatorname{Hom}(T_{n-b}, S)$. Namely,

$$0 \to \operatorname{Hom}(T_0, S) \to \operatorname{Hom}(T_1, S) \to \cdots \to \operatorname{Hom}(T_{n-b}, S)$$

is exact. Since $\tilde{m}^\vee_{n-b} = 0$ by Claim 3.8, it follows by, e.g., [26, Porism 2.2.7], that $\tilde{m}^\vee$ is null-homotopic. ∎

To finalize the proof of Lemma 3.3, it remains to observe that $h_0 \in \mathrm{Hom}(S, Q_1)$ from Claim 3.10 must be zero because h_0 is a homomorphism of graded S-modules and $Q_1 = S(-d)^b$. It follows that $\tilde{m}_1 = d_2 \circ h_1 + h_0 \circ d_1 = d_2 \circ h_1$ and so

$$d_1 \circ \tilde{m}_1 = d_1 \circ d_2 \circ h_1 = 0,$$

as desired. □

Lemma 3.11. *Suppose* $(g_1, \dots, g_n)$ *is a complete intersection ideal of type* $(d)^n$ *in* $k[x_1, \dots, x_n]$ *such that* $(g_{b+1}, \dots, g_n) \subset (x_{b+1}, \dots, x_n)$ *and* $(x_{b+1}, \dots, x_n)^{(n-b)(d-1)+1} \subset (g_{b+1}, \dots, g_n)$*. Then* $g_{b+1}, \dots, g_n \in k[x_{b+1}, \dots, x_n]$*.*

Proof of Lemma 3.11. Since $(x_{b+1}, \dots, x_n)^{(n-b)(d-1)+1} \subset (g_{b+1}, \dots, g_n)$, it follows that

$$g'_j := g_j(0, \dots, 0, x_{b+1}, \dots, x_n), \quad j = b+1, \dots, n,$$

form a regular sequence in $k[x_{b+1}, \dots, x_n]$.

Consider the balanced complete intersection ideals

$$J := (x_1^d, \dots, x_b^d, g_{b+1}, \dots, g_n),$$
$$J' := (x_1^d, \dots, x_b^d, g'_{b+1}, \dots, g'_n),$$

each of which is generated by a regular sequence in S. To establish the lemma, it suffices to show that $J = J'$, which by Corollary 2.4(2) is equivalent to $J_{n(d-1)} = J'_{n(d-1)}$.

Since $\dim_k J_{n(d-1)} = \dim_k J'_{n(d-1)} = \dim_k S_{n(d-1)} - 1$, we only need to prove that $J'_{n(d-1)} \subset J_{n(d-1)}$, and in fact that

$$(g'_{b+1}, \dots, g'_n)_{n(d-1)} \subset J_{n(d-1)}.$$

Fix an element

$$\sum_{i=b+1}^{n} c_i g'_i \in (g'_{b+1}, \dots, g'_n)_{n(d-1)},$$

where c_i's are forms of degree $n(d-1) - d$. Since $x_1^d, \dots, x_b^d \in J$ and

$$k[x_{b+1}, \dots, x_n]_{(n-b)(d-1)+1} \subset J,$$

we can assume that $c_i \in x_1^{d-1} \cdots x_b^{d-1} k[x_{b+1}, \dots, x_n]_{(n-b)(d-1)-d}$. As we have $g'_i \equiv g_i$ (mod $(x_1, \dots, x_b)$), it follows that

$$\sum_{i=b+1}^{n} c_i g'_i \equiv \sum_{i=b+1}^{n} c_i g_i \pmod{(x_1^d, \dots, x_b^d)},$$

which shows that $\sum_{i=b+1}^{n} c_i g'_i \in J$ as required. □

This concludes the proof of Proposition 3.1.

□

4. Preservation of polystability

In this section, we prove Theorem 2.6, which is the main result of this paper. Take $U \in \mathrm{Grass}(n, S_d)_{\mathrm{Res}}$ and let $I := (g \mid g \in U) \subset S$. Since the field k is perfect, we can use the Hilbert-Mumford numerical criterion to analyze the GIT stability of $\mathbf{A}(U)$. So for any non-trivial 1-PS ρ of $\mathrm{SL}(n)$ we choose a basis $x_1, \dots, x_n$ of S_1 on which ρ acts diagonally with weights $w_1 \le w_2 \le \cdots \le w_n$. Note that ρ acts with opposite weights on the dual basis $z_1, \dots, z_n$ of D_1. To apply the numerical criterion to the form $\mathbf{A}(U) \in \mathbb{P}k[z_1, \dots, z_n]_{n(d-1)}$, we observe that by (2.5) a monomial $z_1^{d_1} \cdots z_n^{d_n}$ of degree $n(d-1)$ appears with a non-zero coefficient in $\mathbf{A}(U)$ if and only if $x_1^{d_1} \cdots x_n^{d_n} \notin I_{n(d-1)}$. The following lemma allows us to produce such monomials. Before stating the lemma, recall that $x_1^{a_1} \dots x_n^{a_n} <_{grevlex} x_1^{b_1} \dots x_n^{b_n}$ if and only if either $\sum_{i=1}^n a_i < \sum_{i=1}^n b_i$ or $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ and the last non-zero entry of the vector $(a_1, \dots, a_n) - (b_1, \dots, b_n)$ is positive.

Lemma 4.1 (Grevlex Lemma)**.** *Fix $1 \le a \le n$ and $N \ge 0$. Suppose*

$$\big((x_{a+1}, \dots, x_n)^N\big)_{n(d-1)} \not\subset I_{n(d-1)}.$$

Let $M = x_1^{d_1} \cdots x_n^{d_n}$ be the smallest with respect to $<_{grevlex}$ monomial in $\big((x_{a+1}, \dots, x_n)^N\big)_{n(d-1)}$ that does not lie in $I_{n(d-1)}$. Then for every $i = 1, \dots, a$ we have

$$d_1 + \cdots + d_i \le i(d-1).$$

In particular, taking $a = n$, we conclude that if $M = x_1^{d_1} \cdots x_n^{d_n}$ is the smallest with respect to $<_{grevlex}$ monomial in $k[x_1, \dots, x_n]_{n(d-1)} \setminus I_{n(d-1)}$, then for every $i = 1, \dots, n$ we have

$$d_1 + \cdots + d_i \le i(d-1).$$

Proof. We only use the fact that I is generated in degree d and has codimension n in S. By way of contradiction, suppose that $d_1 + \cdots + d_i > i(d-1)$ for some $i \le a$. Choose a basis $g_1, \dots, g_n$ in U and let $J \subset k[x_1, \dots, x_i]$ be the ideal generated by the forms $g_j(x_1, \dots, x_i, 0, \dots, 0)$, for $j = 1, \dots, n$. Then

$$\dim k[x_1, \dots, x_i]/J = \dim S/(g_1, \dots, g_n, x_{i+1}, \dots, x_n) = 0.$$

Hence by Lemma 2.7, $(k[x_1, \dots, x_i]/J)_K = 0$ for all $K > i(d-1)$. Thus $x_1^{d_1} \cdots x_i^{d_i} \in J$, and so $x_1^{d_1} \cdots x_i^{d_i} \in I + (x_{i+1}, \dots, x_n)$. We conclude that

$$M = x_1^{d_1} \cdots x_i^{d_i} x_{i+1}^{d_{i+1}} \cdots x_n^{d_n} \in I + (x_{i+1}, \dots, x_n) x_{i+1}^{d_{i+1}} \cdots x_n^{d_n}.$$

Since $M \notin I_{n(d-1)}$, there is a monomial $M' \in \big((x_{i+1}, \dots, x_n) x_{i+1}^{d_{i+1}} \cdots x_n^{d_n}\big)_{n(d-1)}$ that does not lie in $I_{n(d-1)}$ either. As $i \le a$, we clearly have $M' \in (x_{a+1}, \dots, x_n)^N$. However, $M' <_{grevlex} M$, which contradicts our choice of M. $\square$

4.0.1. *Proof of semistability.* Let $M = x_1^{d_1}\cdots x_n^{d_n}$ be the smallest with respect to $<_{grevlex}$ monomial of degree $n(d-1)$ that does not lie in $I_{n(d-1)}$. Then $M^\vee := z_1^{d_1}\cdots z_n^{d_n}$ appears with a non-zero coefficient in $\mathbf{A}(U)$. By Lemma 4.1, we have

$$d_1+\cdots+d_i \leq i(d-1), \quad \text{for all } 1\leq i\leq n.$$

Hence the ρ-weight of $z_1^{d_1}\cdots z_n^{d_n}$ satisfies

$$-\sum w_i d_i = \sum_{i=1}^{n}(w_{i+1}-w_i)(d_1+\cdots+d_i) \qquad \text{(here we set } w_{n+1}:=0\text{)}$$
$$\leq \sum_{i=1}^{n}(d-1)i(w_{i+1}-w_i) = -(d-1)\sum_{i=1}^{n} w_i = 0.$$

The Hilbert-Mumford numerical criterion then implies that $\mathbf{A}(U)$ is semistable.

4.0.2. *Proof of polystability: decomposable case.* To prove Theorem 2.6, we proceed by induction on n. The base case is $n=1$, where the statement is obvious because the only balanced complete intersection ideal for $n=1$ is $(x_1^d)\subset k[x_1]$ and the corresponding associated form is z_1^{d-1}, up to a non-zero scalar.

Suppose that the theorem is established for all positive integers less than a given $n\geq 2$ and $U\in \mathrm{Grass}(n,S_d)_{\mathrm{Res}}$ is polystable. If U is decomposable, then by Corollary 2.10, we can assume that for some $1\leq a\leq n-1$, we have a decomposition $U=U_1\oplus U_2$, where $U_1\in\mathrm{Grass}(a,k[x_1,\dots,x_a]_d)_{\mathrm{Res}}$ and $U_2\in\mathrm{Grass}(n-a,k[x_{a+1},\dots,x_n]_d)_{\mathrm{Res}}$. By Lemma 2.11, we have

$$\mathbf{A}(U)=\mathbf{A}(U_1)\mathbf{A}(U_2).$$

Since U_1 and U_2 are polystable with respect to $\mathrm{SL}(a)$ and $\mathrm{SL}(n-a)$ actions, respectively, the induction hypothesis and the following standard result finalizes the proof in the case of a decomposable U:

Lemma 4.2. *Let $V=V_1\oplus V_2$, with $n_i:=\dim V_i\geq 1$. Suppose $F_1\in\mathrm{Sym}^{d_1}V_1$ and $F_2\in\mathrm{Sym}^{d_2}V_2$ are both non-zero, where $n_1d_2=n_2d_1$. Then $F:=F_1F_2$ considered as an element of $\mathrm{Sym}^{d_1+d_2}V$ is $\mathrm{SL}(V)$-polystable if F_i is $\mathrm{SL}(V_i)$-polystable for each i.*

Proof. Let λ be the one-parameter subgroup of $\mathrm{SL}(V)$ such that V_1 is the weight space of λ with weight $-n_2$ and V_2 is the weight space of λ with weight n_1. Then λ stabilizes F by the assumption $n_1d_2=n_2d_1$. The centralizer of λ in $\mathrm{SL}(V)$ is

$$C_{\mathrm{SL}(V)}(\lambda)=\big(\mathrm{GL}(V_1)\times\mathrm{GL}(V_2)\big)\cap\mathrm{SL}(V).$$

Since $\mathrm{char}(k)=0$, [19, Corollary 4.5(a)] (see also [21, Corollaire 2 and Remarque 1]) applies, and so the $\mathrm{SL}(V)$-orbit of F is closed if the $C_{\mathrm{SL}(V)}(\lambda)$-orbit of F is closed. However, by the assumption $n_1d_2=n_2d_1$, every element of the center of $\big(\mathrm{GL}(V_1)\times\mathrm{GL}(V_2)\big)\cap\mathrm{SL}(V)$ acts on F as multiplication by a root of unity. It follows that the $C_{\mathrm{SL}(V)}(\lambda)$-orbit of F is closed if and only if the $\mathrm{SL}(V_1)\times\mathrm{SL}(V_2)$-orbit of F is closed, i.e., if and only if the $\mathrm{SL}(V_i)$-orbit of F_i is closed for $i=1,2$. □

4.0.3. *Proof of polystability: indecomposable case.* Suppose $U \in \mathrm{Grass}(n, S_d)_{\mathrm{Res}}$ is indecomposable. We will use the notation and keep in mind the conclusion of §4.0.1. Assume that for some ρ the limit $\lim_{t\to 0} \rho(t) \cdot \overline{\mathbf{A}(U)}$ exists, where $\overline{\mathbf{A}(U)}$ is a lift of $\mathbf{A}(U)$ to $D_{n(d-1)}$. This implies that $w_\rho(M) = 0$, where $w_\rho(M)$ is the ρ-weight of M. From this we deduce a number of preliminary results.

Lemma 4.3. *Let $1 \le a \le n-1$ be the index such that $w_{a+1} = \cdots = w_n$ and $w_a < w_{a+1}$. Then $d_1 + \cdots + d_a = a(d-1)$.*

Proof. Assuming that $d_1 + \cdots + d_a \le a(d-1) - 1$, we see

$$0 = w_\rho(M) = \sum_{i=1}^{n} d_i w_i > \left(\sum_{i=1}^{a-1} d_i w_i\right) + (d_a+1) w_a + \left(\sum_{i=a+1}^{n} d_i - 1\right) w_n \ge 0,$$

which is impossible. □

Lemma 4.4. *Let a be the integer introduced in Lemma 4.3. Then the homomorphic image of I in $k[x_1, \dots, x_n]/(x_{a+1}, \dots, x_n) \simeq k[x_1, \dots, x_a]$ is a balanced complete intersection ideal of type $(d)^a$ in $k[x_1, \dots, x_a]$.*

Proof. Denote the image ideal by J and suppose that J is not a complete intersection ideal. Then by Lemma 2.7 we have $J_{a(d-1)} = (x_1, \dots, x_a)_{a(d-1)}$, and Lemma 4.3 implies $x_1^{d_1} \cdots x_a^{d_a} \in J_{a(d-1)}$. This means

$$x_1^{d_1} \cdots x_a^{d_a} \in I_{a(d-1)} + (x_{a+1}, \dots, x_n).$$

But then

$$M = x_1^{d_1} \cdots x_a^{d_a} x_{a+1}^{d_{a+1}} \cdots x_n^{d_n} \in I + (x_{a+1}, \dots, x_n) x_{a+1}^{d_{a+1}} \cdots x_n^{d_n}.$$

Since $M \notin I_{n(d-1)}$, there is a monomial

$$M' \in ((x_{a+1}, \dots, x_n) x_{a+1}^{d_{a+1}} \cdots x_n^{d_n})_{n(d-1)}$$

that does not lie in $I_{n(d-1)}$ either. However, $M' <_{grevlex} M$, which is a contradiction. □

Lemma 4.5. *We have*

$$k[x_{a+1}, \dots, x_n]_{(n-a)(d-1)+1} \subset I_{(n-a)(d-1)+1}.$$

Proof. Since I is a homogeneous ideal in S such that S/I a Gorenstein Artin k-algebras of socle degree $n(d-1)$, by Corollary 2.4(1), it suffices to show that $\left((x_{a+1}, \dots, x_n)^{(n-a)(d-1)+1}\right)_{n(d-1)} \subset I_{n(d-1)}$. Assume the opposite and let

$$L = x_1^{c_1} x_2^{c_2} \cdots x_a^{c_a} x_{a+1}^{c_{a+1}} \cdots x_n^{c_n}$$

be the smallest with respect to $<_{grevlex}$ monomial in $\left((x_{a+1}, \dots, x_n)^{(n-a)(d-1)+1}\right)_{n(d-1)}$ that is not in $I_{n(d-1)}$. Then by Lemma 4.1, for every $i \le a$, we have $c_1 + \cdots + c_i \le$

$i(d-1)$. Moreover, $c_1+\cdots+c_a \le a(d-1)-1$ by assumption. But then

$$w_\rho(L) > \sum_{i=1}^{a-1} c_i w_i + (c_a+1)w_a + (n-a)(d-1)w_n \geq 0.$$

As L has positive ρ-weight, it must lie in $I_{n(d-1)}$ by the assumption that the limit $\lim_{t\to 0}\rho(t)\cdot\overline{\mathbf{A}(U)}$ exists, which contradicts our choice of L. □

Lemmas 4.4 and 4.5 imply that both conditions (A) and (B) of Proposition 3.1 are satisfied. Hence U is decomposable, contradicting our assumption. This proves that for every one-parameter subgroup ρ of $\mathrm{SL}(n)$ the limit $\lim_{t\to 0}\rho(t)\cdot\overline{\mathbf{A}(U)}$ does not exist. By the Hilbert-Mumford numerical criterion we then see that $\mathbf{A}(U)$ is polystable. □

We note that our proof in fact gives a more technical version of Theorem 2.6.

Theorem 4.6. *Suppose $U \in \mathrm{Grass}(n, S_d)_{\mathrm{Res}}$. If U is indecomposable, then for every one-parameter subgroup ρ of $\mathrm{SL}(n)$ defined over k the limit $\lim_{t\to 0}\rho(t)\cdot\overline{\mathbf{A}(U)}$ does not exist. In particular, $\mathbf{A}(U)$ is polystable. Furthermore, if U is indecomposable over $\bar{k}$, then $\mathbf{A}(U)$ is stable.*

Notice that, over a non-closed field, the indecomposability of $U \in \mathrm{Grass}(n, S_d)_{\mathrm{Res}}$ does not imply on its own that $\mathbf{A}(U)$ is stable. Indeed, if U is indecomposable over k, it is possible for U to be decomposable over $\bar{k}$. For example,

$$\mathbf{A}\big((x_1+ix_2)^d + (x_1-ix_2)^d, i((x_1+ix_2)^d-(x_1-ix_2)^d)\big) = \langle (z_1^2+z_2^2)^{d-1}\rangle$$

is not stable while the balanced complete intersection is defined and indecomposable over $\mathbb{R}$.

We also note that Theorem 4.6 has a curious consequence for the classification of polystable points in $\mathrm{Grass}(n, S_d)_{\mathrm{Res}}$:

Corollary 4.7. *Suppose $U \in \mathrm{Grass}(n, \mathrm{Sym}^d V)_{\mathrm{Res}}$. Then U is polystable with respect to $\mathrm{SL}(V)$-action if and only if there is a decomposition $V = \bigoplus_{i=1}^m V_i$, where $m \geq 1$ and $n_i := \dim_k V_i \geq 1$, and indecomposable $U_i \in \mathrm{Grass}(n_i, \mathrm{Sym}^d V_i)_{\mathrm{Res}}$ such that*

$$U = \bigoplus_{i=1}^m U_i. \tag{4.8}$$

Proof. Applying Corollary 2.10 repeatedly, we see that a polystable U is of the form given by Equation (4.8) with each U_i indecomposable.

It remains to show that every U with such decomposition is polystable. By Theorem 4.6 each $\mathbf{A}(U_i)$ is polystable, and since $\mathbf{A}(U) = \mathbf{A}(U_1)\cdots\mathbf{A}(U_m)$ by Lemma 2.11, the polystability of $\mathbf{A}(U)$ follows by Lemma 4.2. Since $\mathbf{A}$ is an $\mathrm{SL}(V)$-equivariant locally closed immersion by [2, §2.5], this implies the polystability of U. □

Remark 4.9. It follows from Corollary 4.7 that every $U \in \mathrm{Grass}(n, \mathrm{Sym}^d V)_{\mathrm{Res}}$ whose $\mathrm{SL}(V)$-orbit is not closed in $\mathrm{Grass}(n, \mathrm{Sym}^d V)_{\mathrm{Res}}$ is necessarily decomposable, but cannot be written as a direct sum $\bigoplus_{i=1}^m U_i$, where $V = \bigoplus_{i=1}^m V_i$ and $U_i \in \mathrm{Grass}(\dim V_i, \mathrm{Sym}^d V_i)_{\mathrm{Res}}$ are indecomposable.

5. Invariant-theoretic variant of the Mather-Yau theorem

As before, we continue to work over an arbitrary field k of characteristic 0. Fix $d \geq 2$ and let $(S_{d+1})_\Delta$ be the affine open subset in S_{d+1} of forms defining smooth hypersurfaces in $\mathbb{P}^{n-1}$. An element $F \in (S_{d+1})_\Delta$ defines an isolated homogeneous hypersurface singularity $F(x_1, \ldots, x_n) = 0$. The Jacobian ideal $J_F := (\partial F/\partial x_1, \ldots, \partial F/\partial x_n)$ is a balanced complete intersection ideal, and so the Milnor algebra $M_F := S/J_F$ has a Macaulay inverse system given by the associated form

$$A(F) := \mathbf{A}(\partial F/\partial x_1, \ldots, \partial F/\partial x_n) \in D_{n(d-1)}.$$

The morphism $A\colon (S_{d+1})_\Delta \to D_{n(d-1)}$ gives rise to an $\mathrm{SL}(n)$-contravariant

$$S_{d+1} \to D_{n(d-1)}$$

(see [3] and [17] for details).

We will say that for $F, G \in (S_{d+1})_\Delta$, two singularities $F = 0$ and $G = 0$ are isomorphic if and only if

$$\bar{k}[[x_1, \ldots, x_n]]/(F) \simeq \bar{k}[[x_1, \ldots, x_n]]/(G) \tag{5.1}$$

as algebras over the algebraic closure $\bar{k}$ of k. This condition is equivalent to the existence of a matrix $C \in \mathrm{GL}(n)$, defined over $\bar{k}$, such that $G = C \cdot F$. Indeed, the isomorphism in (5.1) lifts to an automorphism of the power series ring $\bar{k}[[x_1, \ldots, x_n]]$ (see [13, Lemma 1.23]), which is given by a change of variables, and we take C to be its linear part. Note, however, that such a C does not have to exist over k as the example of $F = x_1^4 - x_2^4$ and $G = x_1^4 + x_2^4$ in $\mathbb{R}[x_1, x_2]$ illustrates. Nevertheless, Equation (5.1) is equivalent to the equality of schemes $\mathrm{GL}(n) \cdot F = \mathrm{GL}(n) \cdot G$.

Our results imply that the morphism A sends forms with non-zero discriminant to polystable forms, and from this fact we deduce an invariant-theoretic version of the Mather-Yau theorem (see [23]).

Theorem 5.2. *There exists a finite collection of homogeneous* $\mathrm{SL}(n)$*-invariants* $\mathfrak{I}_1, \ldots, \mathfrak{I}_N$ *on* $D_{n(d-1)}$ *of equal degrees, defined over* k*, such that for any two forms* $F, G \in (S_{d+1})_\Delta$*, the isolated homogeneous hypersurface singularities* $F = 0$ *and* $G = 0$ *are isomorphic if and only if*

$$[\mathfrak{I}_1(A(F)) : \cdots : \mathfrak{I}_N(A(F))] = [\mathfrak{I}_1(A(G)) : \cdots : \mathfrak{I}_N(A(G))].$$

Remark 5.3. Our results show that the Mather-Yau theorem in the homogeneous situation can be extended to the case of an arbitrary field k of characteristic 0 by stating that for $F, G \in (S_{d+1})_\Delta$, the singularities $F = 0$ and $G = 0$ are isomorphic if and only if $M_F \otimes_k \bar{k}$ and $M_G \otimes_k \bar{k}$ are isomorphic as $\bar{k}$-algebras. The main novelty of

Theorem 5.2 is in showing that one can check whether such an isomorphism exists simply by evaluating *finitely many* $\mathrm{SL}(n)$-invariants on the associated forms of M_F and M_G, and that this can be done without passing to the algebraic closure of k.

To prove this result, we will need the following immediate consequence of Theorem 2.6 for the geometry of the associated form morphism (cf. §2.3):

$$\mathbf{A}\colon \operatorname{Grass}(n, S_d)_{\mathrm{Res}} \to \mathbb{P}D_{n(d-1)}.$$

Corollary 5.4. *The induced morphism of GIT quotients*

$$\mathbf{A}/\!/ \mathrm{SL}(n)\colon \operatorname{Grass}(n, S_d)_{\mathrm{Res}}/\!/ \mathrm{SL}(n) \to \mathbb{P}D^{ss}_{n(d-1)}/\!/ \mathrm{SL}(n)$$

is a locally closed immersion.

Proof. By [2, §2.5], $\operatorname{Grass}(n, S_d)_{\mathrm{Res}}$ maps isomorphically via $\mathbf{A}$ to an $\mathrm{SL}(n)$-invariant open subset, say O, of an $\mathrm{SL}(n)$-invariant closed subscheme $Z \hookrightarrow \mathbb{P}D_{n(d-1)}$. By [12, Theorem 1.2], the image of $\mathbf{A}$ lies in the semistable locus Z^{ss}, and Theorem 2.6 implies that O is a saturated open subset of Z^{ss}. The corollary now follows. □

Proof of Theorem 5.2. Note that the two singularities are isomorphic if and only if $\mathrm{GL}(n)\cdot F = \mathrm{GL}(n)\cdot G$ in S_{d+1}, which by the GIT stability of smooth hypersurfaces is equivalent to the fact that F and G map to the same point in the GIT quotient $\mathbb{P}(S_{d+1})_\Delta/\!/ \mathrm{SL}(n)$.

Recall from [2, §2.3] that the (projectivized) morphism $A\colon \mathbb{P}(S_{d+1})_\Delta \to \mathbb{P}D_{n(d-1)}$ factors as the composition of the gradient morphism

$$\nabla\colon \mathbb{P}(S_{d+1})_\Delta \to \operatorname{Grass}(n, S_d)_{\mathrm{Res}},$$

defined by $\nabla F := \langle \partial F/\partial x_1, \dots, \partial F/\partial x_n\rangle$, and the associated form morphism

$$\mathbf{A}\colon \operatorname{Grass}(n, S_d)_{\mathrm{Res}} \to \mathbb{P}D_{n(d-1)}.$$

By Corollary 5.4, the induced morphism of GIT quotients

$$\mathbf{A}/\!/ \mathrm{SL}(n)\colon \operatorname{Grass}(n, S_d)_{\mathrm{Res}}/\!/ \mathrm{SL}(n) \to \mathbb{P}D^{ss}_{n(d-1)}/\!/ \mathrm{SL}(n)$$

is a locally closed immersion.

Next, by [12, Theorem 1.1], we have that $\nabla(F)$ is polystable for every $F \in \mathbb{P}(S_{d+1})_\Delta$. Moreover, by [12, Proposition 2.1 (2)] the induced morphism on the GIT quotients

$$\nabla/\!/ \mathrm{SL}(n)\colon \mathbb{P}(S_{d+1})_\Delta/\!/ \mathrm{SL}(n) \to \operatorname{Grass}(n, S_d)_{\mathrm{Res}}/\!/ \mathrm{SL}(n)$$

is injective. We conclude that the induced morphism

$$A/\!/ \mathrm{SL}(n)\colon \mathbb{P}(S_{d+1})_\Delta/\!/ \mathrm{SL}(n) \to \mathbb{P}D^{ss}_{n(d-1)}/\!/ \mathrm{SL}(n)$$

is injective. The theorem now follows from the definition of the GIT quotient $\mathbb{P}D^{ss}_{n(d-1)}/\!/ \mathrm{SL}(n)$ and the fact that the ring of $\mathrm{SL}(n)$-invariant forms on $D_{n(d-1)}$ is finitely generated by the Gordan-Hilbert theorem. □

Example 5.5. We conclude with an example showing that our GIT stability results are optimal as far as complete intersection Artinian algebras are concerned. Consider the following quasi-homogeneous form:

$$F(x_1, \dots, x_n) = x_1^{d_1+1} + \cdots + x_n^{d_n+1}.$$

Then the Milnor algebra of F is

$$M_F = S/(x_1^{d_1}, \dots, x_n^{d_n}),$$

which is a complete intersection Artin k-algebra of socle degree $(d_1 + \cdots + d_n) - n$. The homogeneous Macaulay inverse system of this algebra (up to a non-zero scalar) is

$$z_1^{d_1-1} \cdots z_n^{d_n-1}.$$

Unless $d_1 = \cdots = d_n$, this form is patently unstable with respect to the $\mathrm{SL}(n)$-action on $D_{(d_1+\cdots+d_n)-n}$, and so all homogeneous $\mathrm{SL}(n)$-invariants vanish on it.

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